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Quasi-periodic solutions for nonlinear wave equations

Solutions quasi périodiques pour l'équation des ondes non linéaire

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ABSTRACT

We construct time quasi-periodic solutions to nonlinear wave equations on the torus in arbitrary dimensions. All previously known results (in the case of zero or a multiplicative potential) seem to be limited to the circle. This extends the method developed in the limit-elliptic setting in [12] to the hyperbolic setting. The additional ingredient is a Diophantine property of algebraic numbers.

RÉSUMÉ

On construit des solutions quasi-périodiques en temps pour l'équation des ondes non linéaire sur le tore en dimension quelconque. Tous les résultats précédents se limitent au cercle. Cet article étend la méthode développée pour le cas limite elliptique dans [12] au cas hyperbolique. Le nouvel ingrédient est une propriété diophantienne des nombres algébriques.

1. Introduction and statement of the Theorem

We consider *real* valued solutions to the nonlinear wave equation (NLW) on the d-torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + v + v^{p+1} + H(x, v) = 0, \tag{1}$$

where $p \in \mathbb{N}$ and $p \geq 1$; considered as a function on $\mathbb{R}^d$, v satisfies: $v(\cdot, x) = v(\cdot, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$; $H(x, v)$ is analytic in x and v and has the expansion:

$$H(x, v) = \sum_{m=p+2}^{\infty} \alpha_m(x) v^m,$$

where α_m as a function on $\mathbb{R}^d$ is $(2\pi)^d$ periodic and real and analytic in a strip of width $\mathcal{O}(1)$ for all m. The integer p in (1) is *arbitrary*.

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Eq. (1) can be rewritten as a first-order equation in t. Let

$$D=\sqrt{-\Delta+1} \tag{2}$$

and

$$u=(v, D^{-1}\frac{\partial v}{\partial t})\in\mathbb{R}^2.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, one obtains the corresponding first-order equation

$$\mathrm{i}\frac{\partial u}{\partial t}=Du+D^{-1}[(\frac{u+\bar{u}}{2})^{p+1}+H(x,\frac{u+\bar{u}}{2})].$$

Using Fourier series, the solutions to the linear equation:

$$\mathrm{i}\frac{\partial u}{\partial t}=Du \tag{3}$$

are linear combinations of eigenfunction solutions of the form:

$$\mathrm{e}^{-\mathrm{i}(\sqrt{j^2+1})t}\mathrm{e}^{\mathrm{i}j\cdot x},\quad j\in\mathbb{Z}^d,$$

where $j^2=|j|^2$ and $\cdot$ is the usual inner product. These solutions are either periodic or quasi-periodic in time.

The main purpose of this note is to announce the following new result, namely, under appropriate genericity conditions on the Fourier frequencies, to be detailed in sect. 2, a class of quasi-periodic solutions to the linear wave equation (3) bifurcates to quasi-periodic solutions of the NLW in (1). We note that when $p\geq\frac{4}{d-2}$ ($d\geq 3$, $H=0$), global solutions to (1) do not seem to be known in general.

Under the assumption that H is a polynomial in u, $\bar{u}$, $\mathrm{e}^{\mathrm{i}x_k}$ and $\mathrm{e}^{-\mathrm{i}x_k}$, $k=1,2,\dots,b$, $x_k\in[0,2\pi)$, below is the precise statement.

Theorem. *Assume that*

$$v^{(0)}(t,x)=\mathrm{Re}\sum_{k=1}^{b}a_k\mathrm{e}^{-\mathrm{i}(\sqrt{j_k^2+1})t}\mathrm{e}^{\mathrm{i}j_k\cdot x}$$

is generic, satisfying the genericity conditions (i–iii), $a=\{a_k\}\in(0,\delta]^b=\mathcal{B}(0,\delta)$ and p even. Assume that $b>C_pd$. Then for all $0<\epsilon<1$, there exists $\delta_0>0$, such that for all $\delta\in(0,\delta_0)$, there is a Cantor set $\mathcal{G}\subset\mathcal{B}(0,\delta)$ with

$$\text{meas}\ \mathcal{G}/\delta^b\geq 1-\epsilon.$$

For all $a\in\mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear wave equation (1):

$$v(t,x)=\mathrm{Re}[\sum a_k\mathrm{e}^{-\mathrm{i}\omega_k t}\mathrm{e}^{\mathrm{i}j_k\cdot x}]+o(\delta^{3/2}),$$

with basic frequencies $\omega=\omega(a)=\{\omega_k(a)\}_{k=1}^b$ satisfying

$$\omega_k=\sqrt{j_k^2+1}+\mathcal{O}(\delta^p),$$

and the amplitude–frequency map $a\mapsto\omega(a)$ is a diffeomorphism. The remainder $o(\delta^{3/2})$ is in a Gevrey norm on $\mathbb{T}^{b+d}$.

Remark. One views quasi-periodic solutions of b frequencies as periodic solutions on a b-dimensional torus in "time". Hence the use of Gevrey norms on $\mathbb{T}^{b+d}$. The condition of large b, namely $b>C_pd$, is imposed in order that certain determinants are not identically zero, as in [12]. It cannot be excluded that this condition could be improved after more technical work. Contrary to [12], however, aside from the genericity conditions, this is the only other condition needed to prove the Theorem. This is because the genericity condition (ii) below dictates that ω is Diophantine, see (4) below; moreover, the mass term 1 in the wave operator D introduces curvature, cf. [11]. The polynomial restriction on H is technical, the result most likely remains valid for analytic H.

This Theorem appears to be the first general existence results on quasi-periodic solutions to the NLW in (1) in arbitrary dimensions. Previously quasi-periodic solutions only seem to have been constructed in one dimension with positive mass m. In that case, the linear wave equation:

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}+mv=0,$$

gives rise to an eigenvalue set $\{\sqrt{j^2+m}, j\in\mathbb{Z}\}$ close to the set of integers, see [4,6,9] and [8,13] in a related context. For almost all m, this set is linearly independent over the integers. This property does not have higher dimensional analogues