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On metric Diophantine approximation in matrices and Lie groups

Approximation diophantienne métrique dans les matrices et les groupes de Lie

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A R T I C L E   I N F O

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A B S T R A C T

We study the Diophantine exponent of analytic submanifolds of $m \times n$ real matrices, answering questions of Beresnevich, Kleinbock, and Margulis. We identify a family of algebraic obstructions to the extremality of such a submanifold, and give a formula for the exponent when the submanifold is algebraic and defined over $\mathbb{Q}$. We then apply these results to the determination of the Diophantine exponent of rational nilpotent Lie groups.

R É S U M É

Nous étudions l'exposant diophantien des sous-variétés analytiques de matrices réelles $m \times n$ et répondons à certaines questions posées par Beresnevich, Kleinbock et Margulis. Nous identifions une famille d'obstructions algébriques à l'extrémalité d'une telle sous-variété, et donnons une formule pour l'exposant lorsque celle-ci est définie sur $\mathbb{Q}$. Enfin, nous appliquons ces résultats à la détermination de l'exposant diophantien des groupes de Lie nilpotents rationnels.

1. Introduction

In their breakthrough paper [11], Kleinbock and Margulis have solved a long-standing conjecture of Sprindzuk regarding metric Diophantine approximation on submanifolds of $\mathbb{R}^n$, stating roughly speaking that non-degenerate submanifolds are extremal in the sense that almost every point on them has similar Diophantine properties to those of a random vector in $\mathbb{R}^n$ (i.e. it is not very well approximable, see below). Doing so, they used new methods coming from dynamics and based on quantitative non-divergence estimates (going back to early work of Margulis [13] and Dani [5]) for certain flows on the

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non-compact homogeneous space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. They suggested at the end of their paper to extend their results to the case of submanifolds of matrices $M_{m,n}(\mathbb{R})$, a natural set-up for such questions. This was studied further in [12,4] and the problem appears in Gorodnik's list of open problems [7].

In this note, we announce a set of results [2] that give a fairly complete picture of what happens in the matrix case as far as extremality is concerned. We identify a natural family of obstructions to extremality (Theorem 4.1) and show that they are in some sense the only obstructions to be considered (Theorem 4.3). Our results also extend to the matrix case the previous works of Kleinbock [8,9] regarding degenerate submanifolds of $\mathbb{R}^n$. When the submanifold is algebraic and defined over $\mathbb{Q}$, we obtain a formula for the exponent (Theorem 5.1).

In a second part of this note, we state new results regarding Diophantine approximation on Lie groups, in the spirit of our earlier work [1]. These results, which are applications of the theorems described in the first part of this note, concern the Diophantine exponent of nilpotent Lie groups and were our initial motivation for studying Diophantine approximation on submanifolds of matrices. The submanifolds to be considered here are images of certain word maps. Depending on the structure of the Lie algebra and its ideal of laws, these submanifolds can be degenerate. The relevant obstructions can nevertheless be identified and this leads to a formula for the Diophantine exponent of an arbitrary rational nilpotent Lie group (Theorem 7.2). A number of examples are also worked out explicitly.

2. Diophantine approximation on submanifolds of $\mathbb{R}^n$

A vector $x \in \mathbb{R}^n$ is called *extremal* (or *not very well approximable*), if for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$|q \cdot x + p| > \frac{c_\varepsilon}{\|q\|^{n+\varepsilon}}$$

for all $p \in \mathbb{Z}$ and all $q \in \mathbb{Z}^n \setminus \{0\}$. Here $q \cdot x$ denotes the standard scalar product in $\mathbb{R}^n$ and $\|q\| := \sqrt{q \cdot q}$ the standard Euclidean norm.

As is well known, almost every $x \in \mathbb{R}^n$ is extremal. An important question in metric Diophantine approximation is that of understanding the Diophantine properties of points x that are allowed to vary inside a fixed submanifold $\mathcal{M}$ of $\mathbb{R}^n$. The submanifold $\mathcal{M}$ is called *extremal* if almost every point on $\mathcal{M}$ is extremal. A key result here is Theorem 2.1.

Theorem 2.1. *(See Kleinbock–Margulis, [11].) Let U be an open connected subset of $\mathbb{R}^k$ and $\mathcal{M} := \{\mathbf{f}(x); x \in U\}$, where $\mathbf{f} : U \to \mathbb{R}^n$ is a real analytic map. Assume that $\mathcal{M}$ is not contained in a proper affine subspace of $\mathbb{R}^n$, then $\mathcal{M}$ is extremal.*

This answered a conjecture of Sprindzuk. The proof made use of homogeneous dynamics via the so-called *Dani correspondence* between Diophantine exponents and the rate of escape to infinity of a diagonal flow in the space of lattices. We will also utilize these tools.

3. Diophantine approximation on submanifolds of matrices

It is natural to generalize this setting to that of submanifolds of matrices, namely submanifolds $\mathcal{M} \subset M_{m,n}(\mathbb{R})$. The Diophantine problem now becomes that of finding good integer approximations (by a vector $p \in \mathbb{Z}^m$) of the image $M \cdot q$ of an integer vector $q \in \mathbb{Z}^n$ under the linear endomorphism $M \in M_{m,n}(\mathbb{R})$. The case $m = 1$ corresponds to the above classical case (that of linear forms), while the dual case $n = 1$ corresponds to a simultaneous approximation.

It turns out that it is more natural to study the slightly more general problem of approximating 0 by the image $M \cdot q$ of an integer vector q. One can pass from the old problem to the new one by embedding $\mathcal{M}$ inside $M_{m,m+n}(\mathbb{R})$, via the embedding (I_m denotes the $m \times m$ identity matrix):

$$M_{m,n}(\mathbb{R}) \to M_{m,m+n}(\mathbb{R})$$

$$M \mapsto (I_m | M)$$

From now on, we will consider an arbitrary connected analytic submanifold $\mathcal{M} \subset M_{m,m+n}(\mathbb{R})$, given as $\mathcal{M} := \{\mathbf{f}(x); x \in U\}$, where $\mathbf{f} : U \to M_{m,m+n}(\mathbb{R})$ is a real analytic map from a connected open subset U in some $\mathbb{R}^k$.

Definition 3.1 *(Diophantine exponent).* We say that a matrix $M \in M_{m+n,n}(\mathbb{R})$ has Diophantine exponent $\beta(M) \geq 0$, if $\beta(M)$ is the supremum of all numbers $\beta \geq 0$ for which there are infinitely many $q \in \mathbb{Z}^{m+n}$ such that

$$\|M \cdot q\| < \frac{1}{\|q\|^\beta}.$$