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On Deligne's periods for tensor product motives

Sur les périodes de Deligne des motifs produits tensoriels

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ABSTRACT

In this paper, we give a description of Deligne's periods $c^{\pm}$ for a tensor product of pure motives $M \otimes M'$ over $\mathbb{Q}$ in terms of the period invariants attached to M and M' by Yoshida [8]. The period relations proved by the author and Raghuram in an earlier paper follow from the results of this paper.

RÉSUMÉ

Nous décrivons dans cette Note les périodes de Deligne $c^{\pm}$ des produits tensoriels $M \otimes M'$ de motifs purs sur $\mathbb{Q}$, en termes des périodes des motifs M et M' et des invariants qui leur sont attachés par Yoshida. Les relations de périodes établies antérieurement par l'auteur et Raghuram résultent de cette description.

1. Introduction

Let M be a pure motive over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(M)$. Suppose that M is critical, then a celebrated conjecture of Deligne [3, Conj. 2.8] relates the critical values of its L-function $L(s, M)$ to certain periods $c^{\pm}(M(\Pi))$, which are defined through a comparison of the Betti and de Rham realizations of the motive.

Conjecturally, one can associate a motive $M(\Pi)$ with a given cohomological cuspidal automorphic representation Π of $GL_n(\mathbb{A}_{\mathbb{Q}})$. One expects from this correspondence that the standard L-function $L(s, \Pi)$ is the motivic L-function $L(s, M(\Pi))$ up to a shift in the s-variable; see Clozel [2, Sect. 4]. There are certain periods $p^{\epsilon}(\Pi)$ that have been defined by Raghuram–Shahidi [7]. Given cohomological cuspidal automorphic representations Π and Σ of $GL_n(\mathbb{A}_{\mathbb{Q}})$ and $GL_{n-1}(\mathbb{A}_{\mathbb{Q}})$, respectively, Raghuram [5,6] has proved that the product $p^{\epsilon}(\Pi)p^{\eta}(\Sigma)$, for a suitable choice of signs ϵ and η, appears in the critical values of the Rankin–Selberg L-function $L(s, \Pi \times \Sigma)$. One can ask whether there is an analogous relation for the Deligne periods so that the results of [6] are compatible with Deligne's conjecture.

In this paper, we give a description of Deligne's periods $c^{\pm}(M \otimes M')$ for the tensor product $M \otimes M'$, where M and M' are two pure motives over $\mathbb{Q}$ all of whose nonzero Hodge numbers are one, in terms of the periods $c^{\pm}(M)$, $c^{\pm}(M')$ *and* some

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other finer invariants attached to M and M' by Yoshida [8]. The main period relations are in Theorems 3.2, 3.4 and 3.6. The period relations for the ratio $\frac{c^+(M\otimes M')}{c^-(M\otimes M')}$ proved by the author and Raghuram in [1] follow from these results.

2. Preliminaries

2.1. Critical motives

Let M be a motive defined over $\mathbb{Q}$ with coefficients in a number field $\mathbb{E}$. Let $H_B(M)$ be the *Betti realization* of M. It is a finite-dimensional vector space over $\mathbb{E}$. The rank $d(M)$ of M is defined to be $\dim_{\mathbb{E}} H_B(M)$. Write $H_B(M) = H_B^+(M) \oplus H_B^-(M)$, where $H_B^{\pm}(M)$ are the ± 1-eigenspaces for the action of complex conjugation ρ on $H_B(M)$. Let $d^{\pm}(M)$ be the $\mathbb{E}$-dimensions of $H_B^{\pm}(M)$. The Betti realization has a *Hodge decomposition*:

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}(M), \tag{2.1}$$

where $H^{p,q}(M)$ is a free $\mathbb{E}\otimes\mathbb{C}$-module of rank $h_M^{p,q}$. The numbers $h_M^{p,q}$ are called the *Hodge numbers* of M. We say that M is *pure* if there is an integer w (which is called the purity weight of M) such that $H^{p,q}(M) = \{0\}$ if $p+q \neq w$. Henceforth, we assume that all the motives we consider are pure. We also have $\rho(H^{p,q}(M)) = H^{q,p}(M)$; and hence ρ acts on the (possibly zero) middle Hodge type $H^{w/2,w/2}(M)$.

Let $H_{DR}(M)$ be the *de Rham realization* of M; it is a $d(M)$-dimensional vector space over $\mathbb{E}$. There is a comparison isomorphism of $\mathbb{E}\otimes_{\mathbb{Q}}\mathbb{C}$-modules:

$$I : H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The de Rham realization has a *Hodge filtration* $F^p(M)$ that is a decreasing filtration of $\mathbb{E}$-subspaces of $H_{DR}(M)$ such that $I(\bigoplus_{p'\geq p} H^{p',w-p'}(M)) = F^p(M)\otimes_{\mathbb{Q}}\mathbb{C}$. Write the Hodge filtration as

$$H_{DR}(M) = F^{p_1}(M) \supsetneq F^{p_2}(M) \supsetneq \cdots \supsetneq F^{p_m}(M) \supsetneq F^{p_{m+1}}(M) = \{0\}; \tag{2.2}$$

all the inclusions are proper and there are no other filtration pieces between two successive members. We assume that the numbers p_μ are maximal among all the choices. Let $s_\mu = h_M^{p_\mu, w-p_\mu}$ for $1 \leq \mu \leq m$. Purity plus the action of complex conjugation on Hodge types says that the numbers p_j and s_μ satisfy $p_j + p_{m+1-j} = w, \forall 1 \leq j \leq m$, and $s_\mu = s_{m+1-\mu}, \forall 1 \leq \mu \leq m$.

We say that the motive M *is critical* if there exist $p^+, p^- \in \mathbb{Z}$ such that $\sum_{i=1}^{p^+} s_i = d^+(M)$ and $\sum_{i=1}^{p^-} s_i = d^-(M)$. In this case, one says that $F^{\pm}(M)$ exists and equals $F^{p^{\pm}}(M)$.

2.2. Tensor product of motives

Let M and M' be pure motives defined over $\mathbb{Q}$ and with coefficients in a number field $\mathbb{E}$. Suppose that their ranks are n and n' and purity weights are w and w', respectively. We further assume that all the non-zero Hodge numbers of M and M' are equal to 1.

Suppose $H_B(M) \otimes \mathbb{C} = \bigoplus_{j=1}^{n} H^{p_j, w-p_j}(M)$, where p_j are integers such that $p_1 < p_2 < \ldots < p_n$. Similarly, suppose $H_B(M') \otimes \mathbb{C} = \bigoplus_{j=1}^{n'} H^{q_j, w'-q_j}(M')$, with $q_1 < q_2 < \ldots < q_{n'}$.

Since all the non-zero Hodge numbers of M and M' are equal to 1, it follows that the Hodge filtrations of the de Rham realizations of M, M' and $M \otimes M'$ are given by

$$\begin{aligned}
H_{DR}(M) &= F^{p_1}(M) \supsetneq F^{p_2}(M) \supsetneq \ldots \supsetneq F^{p_n}(M) \supsetneq (0),\\
H_{DR}(M') &= F^{q_1}(M') \supsetneq F^{q_2}(M') \supsetneq \ldots \supsetneq F^{q_{n'}}(M') \supsetneq (0),\\
H_{DR}(M\otimes M') &= F^{r_1}(M\otimes M') \supsetneq F^{r_2}(M\otimes M') \supsetneq \ldots \supset F^{r_m}(M\otimes M') \supsetneq (0).
\end{aligned}$$

Let u_t denote the dimension of $F^{r_t}(M\otimes M')/F^{r_{t+1}}(M\otimes M')$ for $1 \leq t \leq m$. Let us further assume that $M \otimes M'$ is critical. Consider the complex conjugation action on Betti realizations for the motives M and M'.

If the dimension nn' is an even integer, it follows that $d^{\pm}(M\otimes M')$ are equal to $\frac{nn'}{2}$. From the criticality of $M\otimes M'$, it follows that there is $k^+ = k^- = k_0 \geq 1$ such that

$$u_1 + u_2 + \ldots + u_{k_0} = d^{\pm}(M\otimes M') = \frac{nn'}{2}.$$

Let $1 \leq i \leq n$ and $1 \leq j \leq n'$. Following Yoshida [8], we define:

$$a_i = \left|\{j : 1 \leq j \leq n' : p_i + q_j \leq r_{k_0}\}\right|, \qquad a_j^* = \left|\{i : 1 \leq i \leq n : p_i + q_j \leq r_{k_0}\}\right|.$$