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Complex analysis

Complementability of exponential systems ☆

Complémentabilité des systèmes d'exponentielles

Yurii Belov

Chebyshev Laboratory, St. Petersburg State University, 14th Line 29B, Vasilyevsky Island, St. Petersburg 199178, Russia

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ABSTRACT

We prove that any incomplete systems of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2(-\pi,\pi)$ is a subset of some complete and minimal system of exponentials. In addition, we prove an analogous statement for systems of reproducing kernels in de Branges spaces.

RÉSUMÉ

Nous démontrons que tout système incomplet d'exponentielles complexes $\{e^{i\lambda_n t}\}$ dans $L^2(-\pi,\pi)$ est un sous-ensemble d'un système complet et minimal d'exponentielles. De plus, nous montrons un résultat analogue pour des systèmes de noyaux reproduisants dans les espaces de de Branges.
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1. Introduction

Let a sequence $\Lambda \subset \mathbb{C}$ be such that the system of exponential functions $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is complete and minimal in $L^2(-\pi,\pi)$. Numerous papers are devoted to the different aspects of the theory of exponential systems such as completeness, Riesz basis property, Riesz sequence property, linear summation methods etc. We refer to [9,11,6,5] for details. The geometry of such sequences is not well understood. Nevertheless, there exist a non-geometric necessary and sufficient conditions for the sequence $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ to be complete and minimal; see Theorem 4 in Lecture 18 of [7].

The systems of exponentials are much more regular than an arbitrary system of vectors in $L^2(-\pi,\pi)$. For example, it is easy to verify that if $e^{iat} \in \overline{\operatorname{Span}}\{e^{i\lambda t}\}_{\lambda\in\Lambda}$, $a \notin \Lambda$, then $\overline{\operatorname{Span}}\{e^{i\lambda t}\}_{\lambda\in\Lambda} = L^2(-\pi,\pi)$.

The system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is said to be a Riesz basis if there exists an isomorphism $T : L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ such that $T(e^{int}) = e^{i\lambda_n t} \cdot \|e^{i\lambda_n t}\|^{-1}$, $n \in \mathbb{Z}$ (or, which is the same, $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is an image of an orthogonal basis under a bounded invertible operator). Moreover, $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is said to be a Riesz sequence if it is a Riesz basis of the closure of the space spanned by elements from $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$.

The next question was raised in [8, p. 196]:

Question. *Can every Riesz sequence of complex exponentials be complemented up to a complete and minimal system of complex exponentials?*

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E-mail address: j_b_juri_belov@mail.ru.

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K. Seip has shown in [10, Theorem 2.8] that system $\{e^{\pm i(n+\sqrt{n})t}\}_{n>1}$ is a Riesz sequence of complex exponentials in $L^2(-\pi,\pi)$ that cannot be complemented up to a Riesz basis. We will give a positive answer to the question. Moreover, we do not need the Riesz sequence assumption.

Theorem 1.1. *If $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is an incomplete system in $L^2(-\pi,\pi)$, then there exists a sequence $S\subset\mathbb{R}$, $\Lambda\cap S=\emptyset$ such that the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda\cup S}$ is complete and minimal in $L^2(-\pi,\pi)$.*

In [8], this was proved only for very specific real sequences Λ ($\Lambda=\{n+\delta_n\}_{n\in\mathbb{Z}}$, δ_n is an increasing sequence). It is interesting to note that there exist a lot of complete exponential systems with no complete and minimal subsequences. The simplest example is any sequence $\{\lambda_n\}$ such that $|\lambda_n|\to 0$ (but it may happen even if $|\lambda_n|\to\infty$; see Section 4).

The proof of Theorem 1.1 is nonconstructive and based on the deep and significant de Branges theory of entire functions. In Section 3 we will prove the analogous theorem for the systems of reproducing kernels in *an arbitrary de Branges space*. It would be interesting to find a direct and constructive proof of Theorem 1.1.

In the next section, we transfer our problem to the Paley–Wiener space of entire functions. Section 3.1 is devoted to the de Branges theory. Section 3.2 is devoted to the proof of our result.

2. Paley–Wiener space

We reformulate our problem in the Paley–Wiener space

$$\mathcal{PW}_\pi := \left\{F : F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{itz}dt,\ g\in L^2(-\pi,\pi)\right\}.$$

The Fourier transform $\mathcal{F}$ acts unitarily from $L^2(-\pi,\pi)$ onto $\mathcal{PW}_\pi$; under this transform, the exponential functions become the reproducing kernels in $\mathcal{PW}_\pi$,

$$\mathcal{F}\big(e^{-i\bar\lambda t}\big)=\frac{\sin\pi(z-\bar\lambda)}{\pi(z-\bar\lambda)}=: k_\lambda^{\mathcal{PW}_\pi}(z),$$

$$\big(F,k_\lambda^{\mathcal{PW}_\pi}\big)_{\mathcal{PW}_\pi}=F(\lambda).$$

This gives us the following description (see, e.g., Theorem 4 in Lecture 18 of [7]).

Proposition 2.1. *The system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is complete and minimal in $L^2(-\pi,\pi)$ if and only if the following three conditions hold:*

(i) *the product*

$$G(z):=\lim_{R\to\infty}\prod_{|\lambda|<R}\left(1-\frac{z}{\lambda}\right) \tag{2.1}$$

converges to an entire function G;

(ii) *For some (any) $\lambda\in\Lambda$, we have $\frac{G(z)}{z-\lambda}\in\mathcal{PW}_\pi$;*

(iii) *There is no non-zero entire T such that $GT\in\mathcal{PW}_\pi$.*

If the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is incomplete, then the product (2.1) may diverge. But since Λ satisfies Blaschke's condition $\sum_{\lambda\in\Lambda}\frac{|\mathrm{Im}\,\lambda|}{|\lambda|^2}<\infty$, the product

$$G_\Lambda(z):=\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right)e^{\Re(\lambda^{-1})z} \tag{2.2}$$

converges. Function G_Λ is such that $\frac{G(z)}{G(\bar z)}$ is *a ratio of two Blaschke products* in $\mathbb{C}^+$.

Let $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ be an incomplete system in $L^2(-\pi,\pi)$ and G_Λ be a canonical product from (2.2). Put

$$\mathcal{H}_{\Lambda,\pi}:=\{T : T-\text{ entire, and } G_\Lambda T\in\mathcal{PW}_\pi\}.$$

It is easy to verify that $\mathcal{H}_{\Lambda,\pi}$ is a nontrivial Hilbert space of entire functions (with respect to the norm inherited from the Paley–Wiener space).

Proposition 2.2. *Let $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ be an incomplete system. Then the system $\{e^{i\lambda t}\}_{\lambda\in\Lambda\cup S}$ is complete and minimal in $L^2(-\pi,\pi)$ if and only if the set S is a minimal uniqueness set in $\mathcal{H}_{\Lambda,\pi}$ (i.e. S is a uniqueness set, but $S\setminus\{s_0\}$ is not a uniqueness set for some (any) $s_0\in S$).*

So, it remains to prove that, in $\mathcal{H}_{\Lambda,\pi}$, there exists a minimal uniqueness set.