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Complex analysis

On the class of bi-univalent functions

Sur la classe des fonctions bi-univalentes

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A R T I C L E I N F O

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A B S T R A C T

In an attempt to answer the question raised by A.W. Goodman, we obtain a covering theorem, a distortion theorem, a growth theorem, the radius of convexity and an argument estimate of $f'(z)$ for functions of the class σ of *bi-univalent* functions.

R É S U M É

Dans une tentative de répondre à une question posée par A.W. Goodman, nous obtenons des théorèmes de surjectivité, de déformation et de croissance, ainsi qu'une estimation du rayon de convexité et de l'argument de $f'(z)$ pour une fonction f dans la classe σ des fonctions *bi-univalentes*.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}, \tag{1.1}$$

which are *analytic* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we denote the class of all functions in $\mathcal{A}$ that are *univalent* in $\mathbb{U}$ (for more details on univalent functions, one may refer to [4]).

Obviously, every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in \mathbb{U}$, and $f(f^{-1}(w)) = w$, $|w| < r_0(f)$, $r_0(f) \geq \frac{1}{4}$. Moreover, it is easy to see that the inverse function has the series expansion of the form:

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$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots, \quad w \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, and let σ denote the class of *bi-univalent* functions in $\mathbb{U}$ of the form (1.1). For examples of bi-univalent functions, see the recent work of Srivastava et al. [14], and many other papers like [1,5,8–11,13,15–17].

We emphasize that, as in the class $\mathcal{S}$ of normalized univalent functions, the convex combination of two functions of class σ need not to be bi-univalent. For example, the functions $f_1(z) = \frac{z}{1-z}$ and $f_2(z) = \frac{z}{1+iz}$ are *bi-univalent* but their sum $f_1 + f_2$ is not even univalent, as its derivative vanishes at $\frac{1}{2}(1+i)$. However, the class σ is preserved under a number of elementary transformations. In this regard, we give a result in Section 2.

Lewin [10] investigated the class σ of *bi-univalent functions* and obtained a bound

$$|a_2| < 1.51. \tag{1.2}$$

Motivated by the work of Lewin [10], Brannan and Clunie [2] conjectured that $|a_2| \le \sqrt{2}$. Brannan and Taha [3] introduced the notions of *strongly bi-starlike functions of order* α and *strongly bi-convex functions of order* α and obtained coefficient bounds for $|a_2|$ and $|a_3|$. Following Brannan and Taha [3], many researchers [1,5,8–11,13,15–17] have recently studied several subclasses of σ and obtained coefficient bounds for $|a_2|$ and $|a_3|$.

In a survey article, A.W. Goodman [6, pages 170–172, question number 2] raised the question that $\max|a_n|$, $\max|f'(z)|$, $\max(\arg f'(z))$, etc. are not known for the functions in the class σ.

In the present article, we answer the above question raised by A.W. Goodman [6]. Also, we give the covering theorem for bi-univalent functions, which merely states that the range of each function in the class σ must contain a disk of minimum radius $\frac{1}{3.02}$. Further, we obtain the distortion theorem, the growth theorem and the radius of convexity for the functions of the class σ.

2. Covering theorem for bi-univalent functions

In this section of the paper, first we will show that the class σ is preserved under a number of elementary transformations, and we will give a covering theorem for the class σ. We begin with the partial list of elementary transformations under which the class σ is preserved.

Lemma 2.1. *The class σ is preserved under the following transformations:*

1. ***Rotation**: If $f \in \sigma$, $\theta \in \mathbb{R}$, and $g(z) = e^{-i\theta} f(e^{i\theta}z)$, then $g \in \sigma$;*
2. ***Dilation**: If $f \in \sigma$, $0 < r < 1$, and $g(z) = \frac{1}{r} f(rz)$, then $g \in \sigma$;*
3. ***Conjugation**: If $f \in \sigma$ and $g(z) = \overline{f(\bar{z})}$, then $g \in \sigma$;*
4. ***Disk automorphism**: If $f \in \sigma$, $\zeta \in \mathbb{U}$, and $g(z) = \frac{f(\frac{z+\zeta}{1+\bar{\zeta}z}) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}$, then $g \in \sigma$;*
5. ***Omitted value transformation**: If $f \in \sigma$ with $f(z) \neq w$ for all $z \in \mathbb{U}$, and $g(z) = \frac{wf(z)}{w-f(z)}$, then $g \in \sigma$.*

Proof. The proofs of 1. to 5. are fairly straight forward, and hence we omit the details involved. But for the sake of completeness, we prove the bi-univalency of the omitted value transformation.

In the case of omitted value transformation, the function $g = T \circ f$, with $T(z) = \frac{wz}{w-z}$, where T is a fractional linear transformation, which is univalent and invertible. Since $f \in \sigma$, then $g = T \circ f \in \sigma$, with $g^{-1} = f^{-1} \circ T^{-1}$. □

As the *Koebe function* $f(z) = \frac{z}{(1-z)^2}$ is not a member of the class σ and it plays the role of extremal functions in the class $\mathcal{S}$, the corresponding extremal properties of the class $\mathcal{S}$ is bound to change. As a first result in this direction, we obtain the covering theorem for the class σ. Interestingly, we found that the minimum radius of the disk contained in the range of functions of class σ is increased from $\frac{1}{4}$ to $\frac{1}{3.02}$, which is shown as follows:

Theorem 2.1 *(Covering theorem). The range of every function of the class σ contains the disk $\{w \in \mathbb{C} : |w| \le \frac{1}{3.02}\}$.*

Proof. If $f \in \sigma$ omits the value $w \in \mathbb{C}$, then

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \dots, \quad z \in \mathbb{U},$$

is analytic and bi-univalent in $\mathbb{U}$. Now, combining the inequality (1.2) with $|a_2 + \frac{1}{w}| \le 1.51$, we obtain that $|w| \ge \frac{1}{3.02}$. □

Remarks 2.1. 1. We emphasize that the above property is a necessary condition for a function to be bi-univalent. Also, we note that the famous Koebe function is not bi-univalent, since it does not satisfy the above property. In fact, the maximum of radius of the disk contained in the range of the Koebe function is $\frac{1}{4}$.