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Partial differential equations

Exponential self-similar mixing and loss of regularity for continuity equations

Mélange auto-similaire exponentiel et perte de régularité pour l'équation de continuité

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A R T I C L E   I N F O

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A B S T R A C T

We consider the mixing behavior of the solutions to the continuity equation associated with a divergence-free velocity field. In this Note, we sketch two explicit examples of exponential decay of the mixing scale of the solution, in case of Sobolev velocity fields, thus showing the optimality of known lower bounds. We also describe how to use such examples to construct solutions to the continuity equation with Sobolev but non-Lipschitz velocity field exhibiting instantaneous loss of any fractional Sobolev regularity.

R É S U M É

Nous étudions le comportement de mélange de solutions de l'équation de continuité associée à un champ de vitesse à divergence nulle. Dans cette note, nous décrivons deux exemples explicites de décroissance exponentielle de l'échelle de mélange de la solution. Dans le cas des champs de vitesse Sobolev, nous montrons donc l'optimalité des estimations par dessous connues. Nous décrivons aussi comment utiliser de tels exemples pour construire des solutions de l'équation de continuité à champs de vitesse Sobolev mais non lipschitziens : ces solutions perdent immédiatement toute régularité Sobolev fractionnaire.

1. Introduction

Consider a passive scalar ρ, which is advected by a time-dependent, divergence-free velocity field u on the two-dimensional torus, i.e., a solution ρ to the continuity equation with velocity field u:

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(0,\cdot) = \bar{\rho}, \end{cases} \quad \text{on } \mathbb{R}^+ \times \mathbb{T}^2. \tag{1}$$

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We assume that the initial datum $\bar{\rho}$ satisfies $\int_{\mathbb{T}^2} \bar{\rho} = 0$ (this condition is, at least formally, preserved under the time evolution) and we are interested in the mixing behavior of $\rho(t, \cdot)$ as t tends to $+\infty$.

In order to quantify the level of "mixedness", two different notions of mixing scale are available in the literature. The first one is based on homogeneous negative Sobolev norms (see, for instance, [10,12]), the most common one being the $\dot{H}^{-1}$ norm, which should be viewed as a characteristic length of the mixing in the system (here and in the following, we use the dot as in $\dot{H}^{-1}$ to denote the *homogeneous* Sobolev norm):

Definition 1. The functional mixing scale of $\rho(t, \cdot)$ is $\|\rho(t, \cdot)\|_{\dot{H}^{-1}(\mathbb{T}^2)}$.

The second notion (see [5]) is geometric and has been introduced for solutions with value ± 1:

Definition 2. Given $0 < \kappa < 1/2$, the geometric mixing scale of $\rho(t, \cdot)$ is the infimum $\varepsilon(t)$ of all $\varepsilon > 0$ such that for every $x \in \mathbb{T}^2$, there holds

$$\kappa \le \frac{\mathcal{L}^2(\{\rho(t, \cdot) = 1\} \cap B_\varepsilon(x))}{\mathcal{L}^2(B_\varepsilon(x))} \le 1 - \kappa, \tag{2}$$

where $\mathcal{L}^2$ denotes the two-dimensional Lebesgue measure and $B_\varepsilon(x)$ is the ball of radius ε centered at x.

The parameter κ is fixed and plays a minor role in the definition. Informally, in order for $\rho(t, \cdot)$ to have geometric mixing scale $\varepsilon(t)$, we require that every ball of radius $\varepsilon(t)$ contains a substantial portion of both level sets $\{\rho(t, \cdot) = 1\}$ and $\{\rho(t, \cdot) = -1\}$.

Although, strictly speaking, the two notions above are not equivalent (see the discussion in [11]), they are heuristically very related, and indeed most of the results in this area are available considering any of the two definitions.

The mixing process is studied in the literature under energetic constraints on the velocity field, that is, assuming that the velocity field is bounded with respect to some spatial norm, uniformly in time. We now briefly review some of the related literature (most of the results hold indeed in any space dimension).

(a) The velocity field u is bounded in $\dot{W}^{s,p}(\mathbb{T}^2)$ uniformly in time for some $s < 1$ and $1 \le p \le \infty$ (the case $s = 0$, $p = 2$, often referred to as energy-constrained flow, is of particular interest in applications). In this case, the Cauchy problem for the continuity equation (1) is not uniquely solvable in general (see [1,2]). It is therefore possible to find a velocity field and a bounded solution that is non-zero at the initial time, but is identically zero at some later time. This fact means that it is possible to have perfect mixing in finite time, as already observed in [10] and established in [11] for $s = 0$, building on an example from [7,5].

(b) The velocity field u is bounded in $\dot{W}^{1,p}(\mathbb{T}^2)$ uniformly in time for some $1 \le p \le \infty$ (the case $p = 2$, often referred to as enstrophy-constrained flow, is of particular interest in applications). In this case, the results in [8] ensure uniqueness for the Cauchy problem (1), while for $p > 1$ the quantitative Lipschitz estimates for regular Lagrangian flows in [6] provide an exponential lower bound on the geometric mixing scale, $\varepsilon(t) \ge C \exp(-ct)$. The extension to the borderline case $p = 1$ is still open (see, however, [4]). For the same class of velocity fields, an exponential lower bound for the functional mixing scale, $\|\rho(t, \cdot)\|_{\dot{H}^{-1}} \ge C \exp(-ct)$, has been proved in [9,13].

(c) For velocity fields bounded in $\dot{BV}(\mathbb{T}^2)$ uniformly in time, the results in [3] ensure uniqueness for the Cauchy problem (1). It was recognized in [5] that an exponential decay of the geometric mixing scale can indeed be attained for velocity fields bounded in $\dot{BV}(\mathbb{T}^2)$ uniformly in time, and actually the same example works also for the functional mixing scale.

(d) The velocity field u is bounded in $\dot{W}^{s,p}(\mathbb{T}^2)$ uniformly in time for some $s > 1$ (here the case of interest in applications is that of palenstrophy-constrained flows, that is, $s = 2$ and $p = 2$). In this case, there are no better lower bounds for the decay of the (functional or geometric) mixing scale than the exponential one obtained for $s = 1$. The common belief, supported also by the numerical simulations in [11,9], is that this bound is optimal.

In this note, we sketch two examples in which both the functional and the geometric mixing scales decay exponentially with a velocity field bounded in $\dot{W}^{1,p}(\mathbb{T}^2)$ uniformly in time (for every $1 \le p < \infty$ in the first example, and for every $1 \le p \le \infty$ in the second example, thus including the Lipschitz case). These results show the sharpness of the lower bounds in [6,13,9] (point (b) above), for the full range $1 \le p \le \infty$. Our examples can be seen as a Sobolev (or even Lipschitz) variant of the example in point (c).

Moreover, we describe how such constructions can be employed to obtain counterexamples to the propagation of any fractional Sobolev regularity for solutions to the continuity equation (1) in $\mathbb{R}^d$, when the velocity field belongs to Sobolev classes that do not embed in the Lipschitz space.

After the completion of the present work, we were made aware of a related result obtained independently by Yao and Zlatoš [14], which provides examples of mixing of general initial data, with a rate that is optimal in the range $1 \le p \le \bar{p}$ for some $\bar{p} > 2$.