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Probability theory

# Probabilities of hitting a convex hull

*Probabilités d'atteinte d'une enveloppe convexe*

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ABSTRACT

In this note, we consider the non-negative least-square method with a random matrix. This problem has connections with the probability that the origin is not in the convex hull of many random points. As related problems, suitable estimates are obtained as well on the probability that a small ball does not hit the convex hull.

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RÉSUMÉ

Dans cette Note, nous appliquons la méthode des moindres carrés non négatifs d'une matrice aléatoire. Ce problème est connecté à la probabilité que l'enveloppe convexe de points aléatoires ne contienne pas l'origine. En relation avec ce problème, nous obtenons aussi des estimations de la probabilité qu'une petite boule ne rencontre pas une enveloppe convexe.

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## 1. Introduction

Let $n$ and $m$ be two positive integers with $n \le m$. Suppose that $A$ is an $n \times m$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^n$. In mathematical optimization and other research fields, it is frequent to consider the non-negative least square solution to a linear system $A\mathbf{X} = \mathbf{b}$ with $\mathbf{X} = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m$ under the constraint $\min_{1\le i\le m} x_i \ge 0$. The non-negativity constraints occur naturally in various models involving non-negative data; see [1,4], and [7]. More generally, for non-negative random designs, the matrix $A$ is assumed to be random; see [3] and references therein for this aspect.

The first topic of this note is to investigate the probability $\mathbb{P}\{A\mathbf{X} = \mathbf{b}, \min_{1\le i\le m} x_i \ge 0\}$ when $A$ is a random matrix with suitable restrictions; see Theorem 2.1. The idea of the proof is to change this probability to the one involving the event that the origin is not in the convex hull of many random points, and then apply a well-known result by Wendel [11]. However, instead of applying Wendel's result directly, we propose a new probabilistic proof of it. This probabilistic proof allows us to work on a more general probability of hitting a convex hull by a small ball (instead of the origin) in $\mathbb{R}^n$; see Theorem 4.1.

The study on random convex hulls dates back to 1960s from various perspectives. For instance, in [10] and [2] the expected *perimeter* of a random convex hull was derived. The expected *number of edges* of a random convex hull was

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obtained in [9]. For expected *area* or *volume* of a random convex hull, we refer to [5]. As mentioned earlier, in [11] the probability that the origin does not belong to a random convex hull was perfectly established. In Section 3, we derive an explicit form for the probability that a ball with a small radius $\delta$ in $\mathbb{R}^2$ does not belong to the convex hull of many i.i.d. random points; see Theorem 3.1. This type of probability was considered in [6] together with circle coverage problems. Because of addition assumptions there, unfortunately the results (Corollary 4.2 and Example 4.1) in [6] cannot recover our result (Theorem 3.1 in this note). A more detailed survey on random convex hulls is included in [8].

## 2. A linear system with a random matrix

Since the one-dimension $n = 1$ is trivial, we consider higher dimensions $n \geq 2$. In the proof of the next result, a connection is established between the probabilities of hitting a convex hull and the non-negative solutions to a linear system.

**Theorem 2.1.** *Let $A$ be an $n \times m$, $2 \leq n \leq m$, matrix such that the entries are independent non-negative continuous random variables. Suppose that these random variables have the same mean $\mu$, and are symmetric about the mean. Then the linear system $A\mathbf{X} = (1, 1, \ldots, 1)^T$ has a non-negative solution with probability:*

$$1 - 2^{-m+1} \sum_{k=0}^{n-2} \binom{m-1}{k}.$$

*When $m = n$, it simplifies to $2^{-n+1}$.*

**Proof.** We set the entries of $A$ as $\{a_{ij}\}$, then $\sum_{j=1}^m a_{ij} x_j = 1$ for $1 \leq i \leq n$. Summing over $i$, we obtain $\sum_{j=1}^m (\sum_{i=1}^n a_{ij}) x_j = n$. Let $c_j = \frac{1}{n} \sum_{i=1}^n a_{ij}$, then $\sum_{j=1}^m c_j x_j = 1$. Thus, we can rewrite the linear system $\sum_{j=1}^m a_{ij} x_j = 1$ as $\sum_{j=1}^m (a_{ij} - c_j) x_j = 0$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the column vectors of $A$, and $\mathbf{v} = (1, 1, \ldots, 1)^T$. If we denote $\mathbf{w}_j = \mathbf{a}_j - c_j \mathbf{v}$, then the linear system $\sum_{j=1}^m a_{ij} x_j = 1$ for $1 \leq i \leq n$ has a non-negative solution if and only if there exist $x_1, x_2, \ldots, x_m \geq 0$ with $x_1 + x_2 + \ldots + x_m > 0$ such that $\sum_{j=1}^m x_j \mathbf{w}_j = \mathbf{0}$. In other words, the origin $\mathbf{0}$ belongs to the convex hull of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$. We show that $\{\mathbf{w}_j\}$ are symmetric. Indeed,

$$\begin{aligned}
&\mathbb{P}\big\{\mathbf{w}_j > (t_1, t_2, \ldots, t_n)^T\big\} \\
&\quad = \mathbb{P}\left\{a_{ij} - \frac{1}{n}\sum_{k=1}^n a_{kj} > t_i, 1 \leq i \leq n\right\} = \mathbb{P}\left\{\frac{1}{n}\sum_{k=1}^n (a_{ij} - a_{kj}) > t_i, 1 \leq i \leq n\right\} \\
&\quad = \mathbb{P}\left\{\frac{1}{n}\sum_{k=1}^n \big[(\mu - a_{ij}) - (\mu - a_{kj})\big] > t_i, 1 \leq i \leq n\right\} \\
&\quad = \mathbb{P}\left\{-\frac{1}{n}\sum_{k=1}^n (a_{ij} - a_{kj}) > t_i, 1 \leq i \leq n\right\} = \mathbb{P}\big\{-w_j > (t_1, t_2, \ldots, t_n)^T\big\}.
\end{aligned}$$

Clearly, $\{\mathbf{w}_j\}$ are random vectors in $\mathbb{R}^n$ that lie on the hyperplane $L = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : y_1 + y_2 + \ldots + y_n = 0\}$. Let $p(k, m)$ be the probability that $\mathbf{0}$ does not belong to the convex hull of $m$ symmetric random vectors in $\mathbb{R}^n$ that lie on a $k$-dimensional subspace of $\mathbb{R}^n$. We now compute the probability $p(n-1, m)$. The method below is a probability version of a geometric argument of Wendel [11]. Let $h$ be the indicator function of the event $\mathbf{0} \notin \operatorname{conv}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$. That is, $h(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m) = 1$ if there exists a non-zero vector $\mathbf{b}$ such that $\langle \mathbf{w}_i, \mathbf{b} \rangle \geq 0$ for all $1 \leq i \leq m$, and $h(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m) = 0$ otherwise. Then,

$$p(n-1, m) = \mathbb{P}\big\{\mathbf{0} \notin \operatorname{conv}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)\big\} = \mathbb{E}_{\mathbf{w}} h(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m).$$

Because $\{\mathbf{w}_i\}$ are symmetric, if we let $\{\varepsilon_i\}$ be i.i.d. Bernoulli random variables, then

$$p(n-1, m) = \mathbb{E}_\varepsilon \mathbb{E}_{\mathbf{w}} h(\varepsilon_1 \mathbf{w}_1, \varepsilon_2 \mathbf{w}_2, \ldots, \varepsilon_m \mathbf{w}_m).$$

Noticing that conditioning on $\varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})$, we have

$$\begin{aligned}
p(n-1, m) &= \mathbb{E}_{\varepsilon'} \mathbb{E}_{\mathbf{w}} \mathbb{E}_{\varepsilon m} h(\varepsilon_1 \mathbf{w}_1, \varepsilon_2 \mathbf{w}_2, \ldots, \varepsilon_m \mathbf{w}_m) = \frac{1}{2} \mathbb{E}_{\varepsilon'} \mathbb{E}_{\mathbf{w}} \mathbb{E}_{\varepsilon m} h(\varepsilon_1 \mathbf{w}_1, \varepsilon_2 \mathbf{w}_2, \ldots, \varepsilon_{m-1} \mathbf{w}_{m-1}) \\
&\quad + \frac{1}{2} \mathbb{E}_{\varepsilon'} \mathbb{E}_{\mathbf{w}} \big[2\mathbb{E}_{\varepsilon m} h(\varepsilon_1 \mathbf{w}_1, \varepsilon_2 \mathbf{w}_2, \ldots, \varepsilon_m \mathbf{w}_m) - h(\varepsilon_1 \mathbf{w}_1, \varepsilon_2 \mathbf{w}_2, \ldots, \varepsilon_{m-1} \mathbf{w}_{m-1})\big] \\
&= \frac{1}{2} p(n-1, m-1) + \frac{1}{2} \mathbb{E}_{\varepsilon'} \mathbb{E}_{\mathbf{w}} R
\end{aligned}$$

where

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