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Complex analysis

Bound for the fifth coefficient of certain starlike functions

Borne pour le cinquième coefficient des fonctions étoilées

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A R T I C L E I N F O	A B S T R A C T
	For two different choices of φ, the sharp bound for the fifth coefficient of a normalized analytic function f satisfying $zf'(z)/f(z) \prec \varphi(z)$ is obtained by using a bound for a polynomial in the coefficients of functions with positive real part. Our proof uses a characterization of functions with positive real part in terms of certain positive semi-definite Hermitian form and certain well-known coefficient inequalities for functions with positive real part are shown to follow easily from this characterization. © 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous appuyant sur une majoration de la valeur absolue d'un polynôme en les coefficients de fonctions de partie réelle positive, nous obtenons une majoration précise de la valeur absolue du cinquième coefficient d'une fonction analytique f normalisée, satisfaisant $zf'(z)/f(z) \prec \varphi(z)$, pour deux choix différents de φ. Notre preuve utilise une caractérisation des fonctions de partie réelle positive en termes de certaines formes hermitiennes semi-définies positives. Des inégalités bien connues pour ces fonctions de partie réelle positive résultent aussi sans difficulté de cette caractérisation.

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1. Introduction and preliminaries

Let $\mathcal{S}$ be the class of all normalized univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let φ be an analytic univalent function with positive real part mapping $\mathbb{D}$ onto domains symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. For such φ, Ma and Minda [6] introduced the classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{K}(\varphi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

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The sharp bounds for the first two coefficients of functions in these subclasses have been determined by using the Feketo–Szegö coefficient functional $|a_3 - \mu a_2^2|$ for $\mathcal{K}(\varphi)$ in [6] and the bound for a_4 in [2, Theorem 1, p. 38]. In this paper, our aim is to determine the sharp bound for the fifth coefficient a_5 of functions belonging to the subclasses

$$\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z}) \quad \text{and} \quad \mathcal{S}_{RL}^* := \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right).$$

Sokół [9] conjectured that $|a_n| \leq 1/(2(n-1))$ for $f \in \mathcal{S}_L^*$. Mendiratta et al. [8] conjectured that $|a_n| \leq (5 - 3\sqrt{2})/(2(n-1))$ for $f \in \mathcal{S}_{RL}^*$. They have respectively verified them for $n = 2, 3, 4$. In this paper, we verify these two conjectures for $n = 5$. The proof uses an estimate for a nonlinear coefficient functional for functions with positive real part, which we prove by making use of a characterization of functions with positive real part in terms of certain positive semi-definite Hermitian form. As a consequence of this characterization, certain well-known coefficient inequalities for functions with positive real part are shown to follow easily.

The class $\mathcal{P}$ of functions with positive real part consists of all analytic functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$. For this class, the following lemma related to the Hermitian form is well known (see Ali [1]).

Lemma 1.1. *(See [3].) A function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ if and only if*

$$\sum_{j=0}^{\infty} \left\{ \left| 2z_j + \sum_{k=1}^{\infty} c_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} c_{k+1} z_{k+j} \right|^2 \right\} \geq 0$$

for every sequence $\{z_k\}$ of complex numbers that satisfy $\limsup_{k \to \infty} |z_k|^{1/k} < 1$.

2. Coefficient inequalities for functions with positive real part

By using Lemma 1.1, we first prove the following result.

Lemma 2.1. *Let α, β, γ and a satisfy the inequalities $0 < \alpha < 1$, $0 < a < 1$ and*

$$8a(1-a)((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4\alpha^2(1-\alpha)^2 a(1-a). \tag{2.1}$$

If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then

$$|\gamma c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4| \leq 2.$$

Proof. Let the sequence $\{z_k\}$ of complex numbers be given by $z_0 = \alpha c_3 - \beta c_1 c_2 + \gamma c_1^3$, $z_1 = ac_2 - \delta c_1^2$, $z_2 = bc_1$, $z_3 = -1$, $z_k = 0$ for all $k \geq 4$. For this choice of $\{z_k\}$, Lemma 1.1 gives

$$
\begin{aligned}
&|\gamma c_1^4 + ac_2^2 + (\alpha + b)c_1 c_3 - (\beta + \delta)c_1^2 c_2 - c_4|^2 \\
&\quad \leq |(2\alpha - 1)c_3 + (a + b - 2\beta)c_1 c_2 + (2\gamma - \delta)c_1^3|^2 + |(2a - 1)c_2 + (b - 2\delta)c_1^2|^2 \\
&\qquad - |(a + b)c_1 c_2 - \delta c_1^3 - c_3|^2 + (2b - 1)^2 |c_1|^2 - |bc_1^2 - c_2|^2 + 4 - |c_1|^2 \\
&\quad = 4\alpha(\alpha - 1) \left| c_3 + \left(\frac{\alpha(a+b) + \beta(1-2\alpha)}{2\alpha(\alpha - 1)} \right) c_1 c_2 + \left(\frac{\alpha(\gamma - \delta) + \gamma(\alpha - 1)}{2\alpha(\alpha - 1)} \right) c_1^3 \right|^2 \\
&\qquad + \frac{1}{\alpha(\alpha - 1)} \left| (\alpha\delta - \gamma)c_1^3 + (\alpha(a+b) - \beta)c_1 c_2 \right|^2 + \frac{2}{\alpha(1-\alpha)} (\alpha\delta - \gamma)^2 |c_1|^6 \\
&\qquad + \frac{2}{\alpha(1-\alpha)} (\alpha(a+b) - \beta)^2 |c_1|^2 |c_2|^2 + 4a(a-1) \left| c_2 - \frac{v}{2} c_1^2 \right|^2 + \frac{(\delta - ab)^2}{a(1-a)} |c_1|^4 + 4b(b-1)|c_1|^2 + 4 \\
&\quad \leq \frac{2}{\alpha(1-\alpha)} (\alpha\delta - \gamma)^2 |c_1|^6 + \frac{8}{\alpha(1-\alpha)} (\alpha(a+b) - \beta)^2 |c_1|^2 + \frac{(\delta - ab)^2}{a(1-a)} |c_1|^4 + 4b(b-1)|c_1|^2 + 4,
\end{aligned}
$$

where

$$v = \frac{\delta(1-a) + a(b-\delta)}{a(1-a)}.$$

For $b = \alpha$, $\delta = \beta/2$, the above inequality becomes