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Partial differential equations/Harmonic analysis

Fractional Laplacians, extension problems and Lie groups

Laplaciens fractionnaires, problèmes d'extension et groupes de Lie

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ABSTRACT

We generalize some results concerning the fractional powers of the Laplace operator to the setting of nilpotent Lie Groups and we study its relationship with the solutions to a partial differential equation in the spirit of the articles of Caffarelli & Silvestre [1] and Stinga & Torrea [7].

RÉSUMÉ

Nous généralisons aux groupes de Lie nilpotents les travaux de Caffarelli & Silvestre [1] et Stinga & Torrea [7] concernant la relation existant entre les puissances fractionnaires de l'opérateur laplacien et les solutions d'une équation aux dérivées partielles.

Version française abrégée

Nous présentons ici une extension au cas des groupes de Lie à croissance polynômiale du volume des relations existant entre les solutions $u(t,x)$ de l'équation aux dérivées partielles

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) + \Delta_x u(t,x) = 0, \text{ où } s \in]0,1[, t>0, \text{ et } u(0,x)=\varphi(x), \text{ avec } \varphi \in \mathcal{S}(\mathbb{R}^n),$$

et les puissances fractionnaires du laplacien de la donnée initiale φ. Cette relation est donnée par l'expression

$$\lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x) = -C(s)(-\Delta)^s\varphi(x) \qquad (0<s<1).$$

Étudiée tout d'abord par Caffarelli & Silvestre dans [1], cette relation a été généralisée très rapidement, soit en considérant différentes familles d'opérateurs, soit en travaillant sur des cadres plus généraux (voir [7], [2], [5], [6] et [3]). Nous adoptons ici un point de vue intermédiaire en travaillant sur les groupes de Lie nilpotents : en effet, dans ce cadre, il est intéressant de remarquer qu'il n'existe pas une manière canonique de définir un opérateur laplacien. Nous considérerons alors une famille d'opérateurs laplaciens du type $\mathcal{J} = -\sum_{j=1}^{k} X_j^2$, où les champs de vecteurs invariants à gauche $(X_j)_{1\le j\le k}$ vérifient la condition de Hörmander. Pour ce type de laplaciens, nous démontrons dans cette note le théorème suivant.

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Théorème 0.1. *Soit $\mathbb{G}$ un groupe de Lie nilpotent et soit $\mathcal{J}$ un laplacien qui vérifie la condition de Hörmander. Si $\varphi \in \mathcal{S}(\mathbb{G})$ est une fonction dans la classe de Schwartz, on considère l'équation*

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) - \mathcal{J} u(t,x) = 0,\ pour\ x \in \mathbb{G}\ avec\ s \in]0,1[,\ t > 0\ et\ u(0,x) = \varphi(x).$$

Alors la fonction $u(t,x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} H_\tau \mathcal{J}^s \varphi(x)\, e^{\frac{-t^2}{4\tau}} \frac{d\tau}{\tau^{1-s}}$ *est une solution au sens L^p, avec $1 < p < +\infty$, de l'équation ci-dessus. De plus, on a dans L^p la relation $\mathcal{J}^s \varphi(x) = -C(s) \lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x)$.*

1. Introduction

We are interested in a generalization of the relationship between the solutions $u(t,x)$ of the partial differential equation (also called *extension problem*):

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) + \Delta_x u(t,x) = 0 \text{ with } s \in]0,1[,\ t > 0 \text{ and } u(0,x) = \varphi(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

and the fractional powers of the Laplacian of the initial data φ. This relationship is given by the expression:

$$\lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x) = -C(s)(-\Delta)^s \varphi(x) \qquad (0 < s < 1). \tag{2}$$

This formula was first studied by Caffarelli & Silvestre [1] in 2007 and since then this work has been generalized to many different frameworks (see [7], [2] and [3]). Our aim here is to generalize the relationship between (1) and (2) to the setting of nilpotent Lie groups and in the framework of L^p spaces. In the recent article [2], this relationship is studied in the setting of the Carnot groups and we want to give here a different point of view that is based on the fact that there is not a *unique* way to define a Laplace operator in this framework. Here is an example: the Heisenberg group $\mathbb{H}$ is given by $\mathbb{R}^3$ with the non-commutative group law $x \cdot y = \big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2)\big)$. Associated with this group, we have a Lie algebra $\mathfrak{h}$ given by the left-invariant vector fields $X_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \frac{\partial}{\partial x_3}$ and $T = \frac{\partial}{\partial x_3}$ and we have the identities $[X_1, X_2] = X_1X_2 - X_2X_1 = T$, $[X_i, T] = [T, X_i] = 0$ where $i = 1, 2$. We say that X_1 and X_2 form the *first layer* of the stratification of the Lie algebra $\mathfrak{h}$ while T lies in the *second layer* of the stratification of $\mathfrak{h}$. Now if we want to build from these vector fields an equivalent of the Laplace operator in $\mathbb{H}$, we have several choices:

$$\mathcal{J}_1 = -(X_1^2 + X_2^2), \quad \mathcal{J}_2 = -(X_1^2 + X_2^2 + T), \quad \text{or} \quad \mathcal{J}_3 = -(X_1^2 + X_2^2 + T^2). \tag{3}$$

In [2] the Laplacian was built solely from the vector fields of the *first* layer of the stratification (*i.e.* $\mathcal{J}_1$). In this article we will study this relationship taking into account other types of Laplace operators.

2. Presentation of the framework and statement of the results

Let $\mathbb{G}$ be a connected unimodular Lie group endowed with its Haar measure dx. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$ and consider a family (that will be fixed from now on) of left-invariant vector fields on $\mathbb{G}$

$$\mathbf{X} = \{X_1, \dots, X_k\}, \tag{4}$$

satisfying the *Hörmander condition*: the Lie algebra generated by the X_j for $1 \le j \le k$ is $\mathfrak{g}$. We also have at our disposal the Carnot–Carathéodory metric associated with $\mathbf{X}$, see [8] for a definition. We will denote $\|x\|$ the distance between the origin e and x and $\|y^{-1}\cdot x\|$ the distance between x and y.

For $r > 0$ and $x \in \mathbb{G}$, denote by $B(x,r)$ the open ball with respect to the Carnot–Carathéodory metric centered in x and of radius r, and by $V(r)$ the Haar measure of any ball of radius r. When $0 < r < 1$, there exists $d \in \mathbb{N}^*$, c_l and $C_l > 0$ such that, for all $0 < r < 1$, we have $c_l r^d \le V(r) \le C_l r^d$. The integer d is the *local* dimension of $(\mathbb{G}, \mathbf{X})$. When $r \ge 1$, we will assume that $\mathbb{G}$ is of *polynomial volume growth*, *i.e.* there exist $D \in \mathbb{N}^*$, c_∞ and $C_\infty > 0$ such that, for all $r \ge 1$ we have $c_\infty r^D \le V(r) \le C_\infty r^D$. When $\mathbb{G}$ has polynomial volume growth, the integer D is called the dimension at infinity of $\mathbb{G}$. Recall that nilpotent groups have polynomial volume growth and that a *strict* subclass of the nilpotent groups consists of stratified Lie groups where $d = D$. For example, in the case of the Heisenberg group, we have $d = D = 4$.

Once we have fixed the family $\mathbf{X}$, we define the gradient on $\mathbb{G}$ by $\nabla = (X_1, \dots, X_k)$ and we consider a Laplacian $\mathcal{J}$ on $\mathbb{G}$ defined by

$$\mathcal{J} = -\sum_{j=1}^{k} X_j^2, \tag{5}$$

which is a positive self-adjoint, hypo-elliptic operator since $\mathbf{X}$ satisfies Hörmander's condition, see [8]. Its associated heat operator on $]0, +\infty[\times \mathbb{G}$ is given by $\partial_t + \mathcal{J}$ and we will denote by $(H_t)_{t>0}$ the semi-group obtained from the Laplacian $\mathcal{J}$.