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About a possible analytic approach for walks in the quarter plane with arbitrary big jumps

Autour d'une approche analytique pour les marches à sauts arbitrairement grands dans le quart de plan

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A R T I C L E I N F O

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A B S T R A C T

In this note, we consider random walks in the quarter plane with arbitrary big jumps. We announce the extension to that class of models of the analytic approach of [4], initially valid for walks with small steps in the quarter plane. New technical challenges arise, most of them being tackled in the framework of generalized boundary value problems on compact Riemann surfaces.

R É S U M É

Dans cette note, nous nous intéressons aux marches aléatoires avec sauts arbitrairement grands dans le quart de plan. Nous annonçons le développement, pour cette classe de modèles, de l'approche analytique proposée dans Fayolle et al. (1999) [4], initialement applicable aux marches à petits sauts dans le quart de plan. De nouvelles difficultés théoriques surgissent, qui, pour l'essentiel, sont abordées dans le cadre de la théorie des problèmes aux limites généralisés sur des surfaces de Riemann compactes.

1. Introduction

In the past decades, many fruitful research activities have been dealing with the analysis of random walks in the quarter plane (RWQP) or quadrant $\mathbb{Z}_+^2$. Indeed, these objects are at the crossroads of several domains. In our framework, the initial motivations were twofold: on the one hand, to analyze the stationary distribution of irreducible RWQP [6]; on the other hand, to study queueing models representing two coupled processors working at different service rates [3]. One can also consult [2,4] for modern reference books on analytic and probabilistic aspects of RWQP. Recently, applications were found in enumerative combinatorics, see [1]. Indeed, walks in the quarter plane naturally encode many combinatorial objects (certain trees, maps, permutations, Young tableaux, etc.). They also have many links with population biology and finance. Lastly, in

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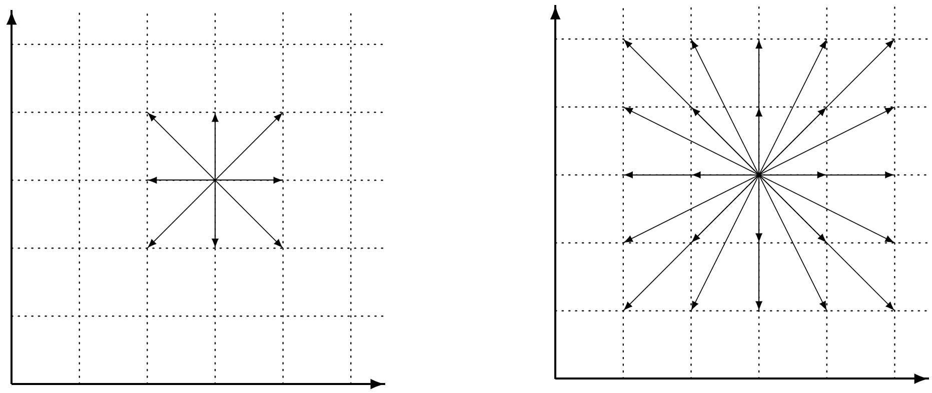

Fig. 1. On the left: walks with small jumps in the quarter plane; on the right: walks with arbitrary big jumps in the quarter plane (of maximal length 2 on the figure).

another context, RWQP can be viewed as particular instances of random processes in cones. This latter topic is the subject of many recent works, due to its links with representation theory, quantum random walks, random matrices, non-colliding random walks, etc.

In most of the analytic studies (in particular in [1,3,4,6]), walks are supposed to have *small steps*. This means that for each unit of time, jumps take place to (a subset of) the eight nearest neighbors on the lattice, see Fig. 1. There are many reasons to make this hypothesis, their fundamental common point being that they simplify the technical computations, and allow one to get closed-form solutions. *In this paper, we aim at presenting some challenges, and we announce results concerning the analysis of random walks having possibly* big jumps *in the quarter plane*. This extension of the analytic approach of [4] has many possible applications, including queueing models, enumeration of lattice paths, discrete harmonic functions, stationary probabilities, etc.

To understand the intrinsic difficulties of such models, we should first recall that the now standard approach (for small steps walks) can be summarized by the three following steps:

(i) find a functional equation for the generating function(s) of interest,
(ii) rewrite the functional equation as a *boundary value problem* (BVP),
(iii) solve the BVP.

This approach concerns several types of problems pertaining to various mathematical areas, and allows one to obtain many quantities: the stationary distribution of ergodic RWQP reflected on the boundary [4,6], the Green functions of killed random walks, the enumeration of deterministic walks [1], fine characteristics of certain queueing systems [3], discrete harmonic functions, etc.

Contrary to point (i), we shall see that points (ii) and (iii) raise more difficult issues, when assumptions on small jumps are relaxed. Indeed, point (i) is rather simple, in the sense that, most of the time, finding a functional equation simply reflects properties of the model. Point (ii), which first appeared in [3], is the keystone of the whole approach. Point (iii) is highly technical, and uses the standard literature (with some peculiarities) devoted to the resolution of BVPs. In many examples, it turns out that the BVPs at stake (ii) have a unique solution, corresponding to the generating functions of interest.

2. Presentation of the model, functional equation, and principles of the approach

2.1. The model

In this section, we introduce the notation, in the particular example of ergodic RWQP, although the announced results can be rendered much more general.

Consider a two-dimensional random process with the following properties.

(P1) The state space is the quarter plane $\mathbb{Z}_+^2 = \{0,1,2,\ldots\}^2$.
(P2) The state space can be represented as the union of non-intersecting classes

$$\mathbb{Z}_+^2 = S \cup \left\{\cup_\ell S'_\ell\right\} \cup \left\{\cup_k S''_k\right\} \cup \left\{\cup_{k,\ell}\left\{(k,\ell)\right\}\right\}. \quad (1)$$

In order to describe the decomposition (1) we need to introduce four parameters, (I^-, J^-) (resp. (I^0, J^0)), which describe the maximal negative amplitude of the transition probabilities in the interior (resp. on the boundary) of the quarter plane. Then, the interior class of the quadrant is