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Stability of electromagnetic cavities perturbed by small perfectly conducting inclusions

Stabilité des cavités électromagnétiques perturbées par des petites inclusions parfaitement conductrices

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we consider an electromagnetic wave propagation problem in harmonic regime in a bounded cavity, in the case where the medium of propagation contains small perfectly conducting inclusions. We prove that the solution to this problem depends continuously on the data in a uniform manner with respect to the size of the inclusions.

R É S U M É

Dans cette note, nous considérons un problème de propagation d'ondes électromagnétiques en régime harmonique dans une cavité bornée, dans le cas où la cavité contient de petites inclusions parfaitement conductrices. Nous montrons que la solution de ce problème dépend continuement des données de manière uniforme vis-à-vis de la taille des inclusions.

Version française abrégée

On considère deux ouverts lipschitziens bornés $\Omega, \mathrm{D} \subset \mathbb{R}^3$ tels que $0 \in \Omega$, et on pose $\Omega_\delta := \{\mathbf{x} \in \Omega,\ \mathbf{x}/\delta \notin \mathrm{D}\}$. On considère également $\omega > 0$ ainsi que deux fonctions à valeurs matricielles $\epsilon, \mu : \Omega \mapsto \mathbb{C}^{3\times 3}$ uniformément bornées pour lesquelles il existe $\epsilon_*, \mu_* > 0$ tels que $\epsilon_* |\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}} \epsilon(\mathbf{x}) \mathbf{y}\}$ et $\mu_* |\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}} \mu(\mathbf{x}) \mathbf{y}\}$ pour tout $\mathbf{x} \in \Omega, \mathbf{y} \in \mathbb{R}^3$. Sous l'hypothèse que ω n'est pas une fréquence de résonance du problème de Maxwell dans Ω avec condition de conducteur parfait sur le bord, on démontre (voir Théorème 2.1) qu'il existe deux constantes $C, \delta_0 > 0$ indépendantes de δ telles que, pour tout $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega_\delta) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial\Omega_\delta\}$, on a :

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$$\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)} \leq C \sup_{\mathbf{v}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\setminus\{0\}} \frac{\int_{\Omega_\delta} \mu^{-1}\mathbf{curl}(\mathbf{u})\mathbf{curl}(\mathbf{v}) - \omega^2(\epsilon\mathbf{u})\mathbf{v}\,\mathrm{d}x}{\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)}} \quad \forall\delta\in[0,\delta_0].$$

0. Introduction

In this note, $\Omega, \mathrm{D} \subset \mathbb{R}^3$ refer to bounded Lipschitz open sets with $0 \in \Omega$. Set $\Omega_\delta := \{\mathbf{x}\in\Omega,\ \mathbf{x}/\delta \notin \mathrm{D}\}$ and $\Xi = \mathbb{R}^3 \setminus \overline{\mathrm{D}}$. Let $\epsilon, \mu : \Omega \mapsto \mathbb{C}^{3\times 3}$ refer to bounded matrix valued functions such that there exist constants $\epsilon_*, \mu_* > 0$ with $\epsilon_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\epsilon(\mathbf{x})\mathbf{y}\}$ and $\mu_*|\mathbf{y}|^2 < \Re e\{\mathbf{y}^{\mathrm{T}}\mu(\mathbf{x})\mathbf{y}\}$ for all $\mathbf{x}\in\Omega, \mathbf{y}\in\mathbb{R}^3$. The matrices ϵ and μ stand respectively for the electric permittivity and the magnetic permeability of the medium. We also consider a fixed frequency $\omega > 0$, and study the corresponding Maxwell source problem

$$\begin{cases} \mathbf{u}_\delta \in \mathbf{H}_0(\mathrm{curl}, \Omega_\delta) \quad \text{such that} \\ \mathbf{curl}^2_\mu(\mathbf{u}_\delta) - \omega^2\epsilon\mathbf{u}_\delta = \mathbf{f} \quad \text{in } \Omega_\delta, \end{cases} \tag{1}$$

where $\mathbf{H}_0(\mathrm{curl}, \Omega_\delta) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\Omega_\delta),\ \mathbf{u}\times\mathbf{n} = 0 \text{ on } \partial\Omega_\delta\}$, and $\mathbf{curl}^2_\mu := \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\ \cdot)$. Here $\mathbf{f} \in \mathbf{H}_0(\mathrm{curl}, \Omega_\delta)^*$ is an arbitrarily chosen right hand side. It is well established that, for any fixed $\delta > 0$, the problem above is of Fredholm type with index 0, i.e. it admits a unique solution except for a discrete set of eigenfrequencies, see, e.g., [7]. In the present note, we will assume that ω is not an eigenfrequency of the limit problem associated with $\delta = 0$ (where $\Omega_0 = \Omega$), and we wish to show that, for $\mathbf{f}$ fixed, the solution remains uniformly bounded as $\delta \to 0$.

Although this kind of asymptotic stability result is well known for many different situations in the case of scalar elliptic problems (see [8–14] for example), Maxwell's equations have received much less attention.

A series of works [2,1,3,6] deals with electromagnetic scattering in homogeneous ambient media containing small penetrable heterogeneities. In such situations, the propagation medium is fixed, i.e. the perturbed problem is posed in the same domain as the limit problem. On the other hand, the contributions [4] and [5, Chap. 3] deal with the asymptotics of perfectly conducting (impenetrable) objects embedded in a homogeneous medium by means of boundary integral equation techniques. It provides a stability estimate for a second kind integral equation (the so-called MFIE), and deduces asymptotic formulas for the electromagnetic fields in a region of the domain located at a fixed positive distance from the small scatterers. The analysis in [4,5] strongly relies on boundary integral representation formulas, adopting a very different approach compared to [2] as regards stability.

Adapting the proof of stability contained in [2] to the case of perfectly conducting inclusions seems difficult at first sight because, in this case, the medium of propagation varies as $\delta \to 0$, which prevents the use of compact embedding theorems that play a key role in the approach of [2]. This is the goal of the present note to show how to circumvent this difficulty by means of an asymptotic version of Hardy's inequality.

1. Asymptotic Hardy inequality

In the sequel, for any bounded Lipschitz open set $\mathcal{O} \subset \mathbb{R}^3$, the space $\mathrm{L}^2(\mathcal{O})$, resp. $\mathbf{L}^2(\mathcal{O}) := \mathrm{L}^2(\mathcal{O})^3$, will refer to square integrable functions, resp. fields, equipped with the norm $\|v\|^2_{\mathrm{L}^2(\mathcal{O})} := \int_{\mathcal{O}} |v|^2\,\mathrm{d}\mathbf{x}$. Moreover we consider the spaces $\mathbf{H}(\mathrm{curl}, \mathcal{O}) := \{\mathbf{u} \in \mathbf{L}^2(\mathcal{O}),\ \mathbf{curl}(\mathbf{u}) \in \mathbf{L}^2(\mathcal{O})\}$, and $\mathbf{H}_0(\mathrm{curl}, \mathcal{O}) := \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}, \mathcal{O}),\ \mathbf{u}\times\mathbf{n} = 0\}$ equipped with the norm $\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl},\mathcal{O})} = \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathbf{curl}(\mathbf{v})\|^2_{\mathbf{L}^2(\mathcal{O})}$, where $\mathbf{n}$ refers to the normal vector to $\partial\mathcal{O}$. In addition, $\mathbf{X}(\mathcal{O})$ will refer to the fields $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \mathcal{O})$ such that $\mathrm{div}(\epsilon\mathbf{v}) \in \mathrm{L}^2(\mathcal{O})$ and $\int_\Sigma \epsilon\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma = 0$ for each connected component Σ of $\partial\mathcal{O}$, equipped with the norm

$$\|\mathbf{v}\|^2_{\mathbf{X}(\mathcal{O})} = \big\|\mathbf{curl}(\mathbf{v})\big\|^2_{\mathbf{L}^2(\mathcal{O})} + \big\|\mathrm{div}(\epsilon\mathbf{v})\big\|^2_{\mathrm{L}^2(\mathcal{O})} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})}.$$

The space $\mathbf{X}(\mathcal{O})$ is compactly embedded into $\mathbf{L}^2(\mathcal{O})$, see [17].

Lemma 1.1. *There exist constants $C, \delta_0 > 0$ independent of δ such that*

$$C\int_{\Omega_\delta} \frac{|\mathbf{v}(\mathbf{x})|^2}{\delta^2 + |\mathbf{x}|^2}\,\mathrm{d}\mathbf{x} \leq \|\mathbf{v}\|^2_{\mathbf{X}(\Omega_\delta)} \quad \forall\mathbf{v}\in\mathbf{X}(\Omega_\delta),\ \forall\delta\in(0,\delta_0).$$

Proof. Let $\mathbf{W}(\Xi)$ refer to the closure of $\mathscr{C}^\infty_{\mathrm{comp}}(\overline{\Xi}) = \{\mathbf{v}|_\Xi,\ \mathbf{v}\in\mathscr{C}^\infty(\mathbb{R}^3)^3,\ \mathrm{supp}(\mathbf{v}) \text{ bounded}\}$ with respect to the norm $\|\mathbf{v}\|^2_{\mathbf{W}(\Xi)} := \int_\Xi (1+|\xi|^2)^{-1}|\mathbf{v}(\xi)|^2\,\mathrm{d}\xi + \|\mathbf{curl}(\mathbf{v})\|^2_{\mathbf{L}^2(\Xi)} + \|\mathrm{div}(\mathbf{v})\|^2_{\mathbf{L}^2(\Xi)}$. Set in addition $\mathbf{W}_0(\Xi) = \{\mathbf{v}\in\mathbf{W}(\Xi),\ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\Xi\}$, which is a closed subspace of $\mathbf{W}(\Xi)$. According to Lemma 6 in [15], if Γ_j, $j = 0, 1, \ldots, J$ refer to the connected components of $\partial\Xi$, there exists $C > 0$ such that

$$C\int_\Xi \frac{|\mathbf{w}(\xi)|^2\,\mathrm{d}\xi}{1+|\xi|^2} \leq \big\|\mathbf{curl}(\mathbf{w})\big\|_{\mathbf{L}^2(\Xi)} + \big\|\mathrm{div}(\mathbf{w})\big\|_{\mathbf{L}^2(\Xi)} + \sum_{j=1}^{J}\left|\int_{\Gamma_j} \mathbf{w}\cdot\mathbf{n}\,\mathrm{d}\sigma\right| \quad \forall\mathbf{w}\in\mathbf{W}_0(\Xi). \tag{2}$$