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Noncommutative affine spaces and Lie-complete rings

Espaces affines non commutatifs et anneaux de Lie complets

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ABSTRACT

In this paper, we investigate the structure sheaves of an (infinite-dimensional) affine NC-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, affine Lie-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$, and their nilpotent perturbations $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}$, respectively. We prove that the schemes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$ are identical if and only if $\mathbf{x}$ is a finite set of variables, that is, when we deal with finite-dimensional noncommutative affine spaces. For each (Zariski) open subset $U \subseteq X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$, we obtain the precise descriptions of the algebras $\mathcal{O}_{\mathfrak{nc}}(U)$, $\mathcal{O}_{\mathfrak{nc},q}(U)$, $\mathcal{O}_{\mathfrak{lieh},q}(U)$ and $\mathcal{O}_{\mathfrak{lieh},q}(U)$ of noncommutative regular functions on U associated with the schemes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$, respectively. The obtained result for $\mathcal{O}_{\mathfrak{nc}}(U)$ generalizes Kapranov's formula in the finite-dimensional case. Our approach to the matter is based on a noncommutative holomorphic functional calculus in Fréchet algebras.

RÉSUMÉ

Dans cette note, nous étudions la structure des faisceaux des NC-espaces $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ et des Lie espaces $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$, affines (de dimension infinie), et de leur perturbations nilpotentes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}$, respectivement. Nous montrons que les schémas $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$ sont identiques si et seulement si x est un ensemble fini de variables, c'est-à-dire lorsqu'on traite des espaces affines non commutatifs de dimension finie. Pour chaque ouvert (de Zariski) $U \subset X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$, nous obtenons les descriptions précises des algèbres $\mathcal{O}_{\mathfrak{nc}}(U)$, $\mathcal{O}_{\mathfrak{nc},q}(U)$, $\mathcal{O}_{\mathfrak{lieh}}(U)$ et $\mathcal{O}_{\mathfrak{lieh},q}(U)$, de fonctions régulières non commutatives sur U, associées aux schémas $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}$, respectivement. Ces résultats pour $\mathcal{O}_{\mathfrak{nc}}(U)$ généralisent la formule de Kapranov dans le cas où la dimension est finie. De plus, nous montrons que tout anneau Lie complet A est plongé dans $\Gamma(X, \mathcal{O}_A)$ comme sous-algèbre dense pour la topologie I_1-adique associée à l'idéal bilatère I_1 engendré par tous les commutateurs de A.

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1. Introduction

The main idea of scheme-theoretic algebraic geometry is the duality correspondence between commutative rings and affine schemes [10,11]. Based on this duality, noncommutative affine schemes are defined as the dual of the category of associative rings [13,15]. The affine NC-schemes are defined as noncommutative nilpotent thickenings of commutative schemes due to Kapranov [12]. If A is a noncommutative associative algebra with its commutativization $A_c = A/\mathcal{I}([A,A])$, then the surjective homomorphism $A \to A_c$ allows us to embed the geometric object $X = \mathrm{Spec}(A_c)$ into an affine NC-scheme $(X, \mathcal{O}_A)$, which is a ringed space equipped with a noncommutative structure sheaf $\mathcal{O}_A$ of NC-complete algebras. Recall that an associative (complex) algebra A can be equipped with an NC-topology defined by the commutator filtration $(F^k(A))_k$, where $F^k(A) = \sum_m \sum_{i_1+\cdots+i_m=k} I_{i_1}\cdots I_{i_m}$ and $I_s = \mathcal{I}(A_{\mathfrak{lie}}^{(s+1)})$ is the two-sided ideal in A generated by the $(s+1)$-th member of the lower central series $A_{\mathfrak{lie}}^{(s+1)}$ of the related Lie algebra $A_{\mathfrak{lie}}$. The algebra A is called an NC-complete algebra if it is Hausdorff and complete with respect to the NC-topology in A. The formal spectrum $X = \mathrm{Spf}(A)$ of an NC-complete algebra A is reduced to $\mathrm{Spf}(A_c)$, and the structure sheaf $\mathcal{O}_A$ is defined as the sheaf of continuous sections of the covering space over X defined by the noncommutative topological localizations of A [12]. In particular, the affine NC-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ (over the complex field) is defined as the formal scheme $\mathrm{Spf}(\mathcal{O}_{\mathfrak{nc}}(\mathbf{x}))$ of the NC-completion $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x})$ of the free associative algebra $\mathbb{C}\langle\mathbf{x}\rangle$ in the independent variables $\mathbf{x} = (x_i)_{i\in\Xi}$, whose structure sheaf is denoted by $\mathcal{O}_{\mathfrak{nc}}$.

The formal schemes can be constructed for Lie-complete rings either. Recall that a ring A is said to be a Lie-nilpotent ring if $A_{\mathfrak{lie}}$ is a nilpotent Lie ring. A Lie-complete ring A is defined as a complete filtered ring associated with a filtration $(J_\alpha)_\alpha$ whose quotients A/J_α are Lie-nilpotent rings. They admit topological localizations that are commutative modulo their topological nilradicals (see below Proposition 1.1). The free algebra $\mathbb{C}\langle\mathbf{x}\rangle$ admits various completions that are Lie-complete algebras. First consider the free Lie-nilpotent algebra $B_q(\mathbf{x}) = \mathbb{C}\langle\mathbf{x}\rangle/I_q$ of index q, which is the Hausdorff completion of $\mathbb{C}\langle\mathbf{x}\rangle$ with respect to the filtered topology of the (singleton) filtration (I_q). We have also its I_1-adic (or NC) completion $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$, which is the Hausdorff completion of $\mathbb{C}\langle\mathbf{x}\rangle$ defined by one of the equivalent filtrations $(I_q + I_1^k)_k$ and $(I_q + F_k)_k$, where $F_k = F_k(\mathbb{C}\langle\mathbf{x}\rangle)$. Actually, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$ is the NC-completion of $B_q(\mathbf{x})_{\mathfrak{h}} = \mathbb{C}\langle\mathbf{x}\rangle/\overline{I_q}^{\mathfrak{nc}}$, where $\overline{I_q}^{\mathfrak{nc}} = \bigcap_m (I_q + F_m)$ is the NC-closure of I_q in $\mathbb{C}\langle\mathbf{x}\rangle$. The completion of $\mathbb{C}\langle\mathbf{x}\rangle$ with respect to the filtration $(\overline{I_k}^{\mathfrak{nc}})_k$ is denoted by $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x})$, whereas $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$ denotes the completion of $\mathbb{C}\langle\mathbf{x}\rangle$ associated with $(I_k)_k$. Since $\mathbb{C}\langle\mathbf{x}\rangle = \mathcal{U}(\mathfrak{L}(\mathbf{x}))$ is the universal enveloping algebra of the free Lie algebra $\mathfrak{L}(\mathbf{x})$ generated by $\mathbf{x}$, we have the two-sided ideal $\mathfrak{J}_q = \mathcal{I}(\mathfrak{L}(\mathbf{x})_{\mathfrak{lie}}^{(q+1)})$ in $\mathbb{C}\langle\mathbf{x}\rangle$. The Hausdorff completion $\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})$ of $\mathbb{C}\langle\mathbf{x}\rangle$ is defined by one of the equivalent filtrations $(\mathfrak{J}_q + I_1^k)_k$ and $(\mathfrak{J}_q + F_k)_k$. Note that it is just I_1-adic completion of $\mathcal{U}(\mathfrak{g}_q(\mathbf{x}))$, where $\mathfrak{g}_q(\mathbf{x}) = \mathfrak{L}(\mathbf{x})/\mathfrak{L}(\mathbf{x})_{\mathfrak{lie}}^{(q+1)}$ is the free nilpotent Lie algebra of index q generated by $\mathbf{x}$. Thus we have the Lie-complete algebras $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x})$, $\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})$, $B_q(\mathbf{x})$, $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x})$. Note that $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x}) = \varprojlim\{\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})\} = \varprojlim\{\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})\}$, $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x}) = \varprojlim\{B_q(\mathbf{x})\}$, and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x}) = \varprojlim\{B_q(\mathbf{x})_{\mathfrak{h}}\}$ up to the topological isomorphisms. The structure sheaves defined by these Lie-complete algebras are denoted by $\mathcal{O}_{\mathfrak{nc}}$, $\mathcal{O}_{\mathfrak{nc},q}$, B_q, $\mathcal{O}_{\mathfrak{lie}}$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}$, respectively, and they in turn generate the schemes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h},q}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$, called noncommutative affine spaces. Note that the (topological) commutativizations of these algebras are reduced to $\mathbb{C}[\mathbf{x}]$ and their formal spectra are reduced to $X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$ equipped with the Zariski topology. The identity mapping over X generates the scheme morphisms $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}} \to \mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}} \to \mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$. In the finite-dimensional case, these morphisms are identical, that is, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}} = \mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}} = \mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ iff $\mathrm{Card}(\mathbf{x}) < \infty$. But in the infinite-dimensional case, we have different structure sheaves $\mathcal{O}_{\mathfrak{nc}}$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}$ over X. This is a new phenomenon that appeared in the infinite-dimensional case having the affine Lie-space apart form the affine NC-space. Similar situation takes place for their q-versions.

In the present note, we propose descriptions of the structure sheaves associated with these noncommutative affine spaces. Our approach to the matter is based on the formally-radical holomorphic functions $\mathcal{F}_{\mathfrak{g}}(U)$ in elements of a nilpotent Lie algebra $\mathfrak{g}$ developed in [2,3] (see also [1]). The Fréchet algebras of noncommutative holomorphic functions have been developed to implement Taylor's program on the noncommutative holomorphic functional calculus for operator families generating a nilpotent Lie algebra [3–7].

2. The structure sheaf of a Lie-complete ring

Let A be a filtered ring with its filtration $\mathfrak{a} = (J_\alpha)_\alpha$, $S \subseteq A\backslash\{0\}$ a (topologically) closed and multiplicatively closed subset in A satisfying the following *topological right* (similarly, *left*) *Ore conditions:*

($TR1$) for $s \in S$ and $a \in A$ there exist nets $(t_\iota) \subseteq S$ and $(b_\iota) \subseteq A$ such that $\lim_\iota (sb_\iota - at_\iota) = 0$;

($TR2$) if $sa \in J_\alpha$ with $s \in S$ and $a \in A$, then $at \in J_\alpha$ for some $t \in S$.

Then A admits the topological localization $A[S^{-1}]$ of right (respectively, left) fractions, which is a complete filtered ring with its continuous ring homomorphism $\varphi : A \to A[S^{-1}]$ such that $\varphi(S) \subseteq A[S^{-1}]^*$ (consists of units) and $\{\varphi(a)\varphi(s)^{-1} : a \in A, s \in S\}$ is dense in $A[S^{-1}]$. The filtered ring $A[S^{-1}]$ possesses the following universal property. If $\psi : A \to B$ is a continuous ring homomorphism into another complete filtered ring B such that $\psi(S) \subseteq B^*$, then there exists a unique continuous ring homomorphism $\sigma : A[S^{-1}] \to B$ such that $\sigma \cdot \varphi = \psi$.

Now let A be a Lie-complete ring with its topological nilradical $\mathfrak{Tnil}(A) = \{a \in A : \lim_n a^n = 0\}$, and $X = \mathrm{Spf}(A)$. Then $X = \mathrm{Spf}(A_c)$ is a topological space equipped with a Zariski topology, and $\mathfrak{Tnil}(A) = \bigcap\{\mathfrak{p} : \mathfrak{p} \in \mathrm{Spf}(A)\}$, where $A_c = A/\overline{I_1}$. If I_1 is open (in particular, $\mathfrak{Tnil}(A)$ is open), then $X = \mathrm{Spec}(A_c)$. Note that I_1 is open for an NC-complete ring A.