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Differential geometry

Rigidity in a conformal class of contact form on CR manifold

Rigidité dans une classe conforme de formes de contact sur une variété CR

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, we first prove that any two conformal contact forms on a compact CR manifold that have the same pseudo-Hermitian Ricci curvature must be different by a constant. In another direction, we prove a CR analogue of the conformal Schwarz lemma of Riemannian geometry.

R É S U M É

Dans cet article, nous montrons d'abord que deux formes de contact conformes quelconques sur une variété compacte CR qui ont la même courbure de Ricci pseudo-hermitienne ne diffèrent que d'un facteur constant. Dans une autre direction, nous prouvons un analogue CR du lemme de Schwarz conforme de la géométrie riemannienne.

1. Introduction

In this paper, we are going to prove some rigidity results in CR geometry. First, we recall the following result of Xu in [8]:

Theorem 1.1. *Suppose (M, g) is a compact Riemannian manifold without boundary of dimension ≥ 2. If $\tilde{g} = \mathrm{e}^{2u} g$ such that their Ricci curvatures satisfy $\mathrm{Ric}(\tilde{g}) = \mathrm{Ric}(g)$, then u is a constant.*

We will prove the CR analog of Theorem 1.1. More precisely, we prove the following:

Theorem 1.2. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n + 1$ with a given contact form θ. If $\tilde{\theta} = \mathrm{e}^{2u}\theta$ is such that their pseudo-Hermitian Ricci curvatures satisfy $\mathrm{Ric}(\tilde{\theta}) = \mathrm{Ric}(\theta)$, then u is a constant.*

In another direction, we recall the following conformal Schwarz lemma, which was first proved by Yau [9]:

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Theorem 1.3. *Suppose (M, g) is a compact Riemannian manifold without boundary of dimension ≥ 2 whose scalar curvature satisfies $R_g \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$, and g_Y is the Yamabe metric conformally equivalent to g with scalar curvature $R_{g_Y} = -1$. Then we have*

$$\frac{g_Y}{|R_{\min}|} \leq g \leq \frac{g_Y}{|R_{\max}|}.$$

In [7], Suárez-Serrato and Tapie used the Yamabe-type flow to reprove Theorem 1.3. Using the CR Yamabe-type flow, we will prove the following CR analog of Theorem 1.3:

Theorem 1.4. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies $R_\theta \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$, and θ_Y is the contact form conformally equivalent to θ with Webster scalar curvature $R_{\theta_Y} = -1$. Then we have:*

$$\frac{\theta_Y}{|R_{\min}|} \leq \theta \leq \frac{\theta_Y}{|R_{\max}|}. \tag{1.1}$$

As a corollary, we have the following:

Corollary 1.5. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies $R_\theta \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$. Then we have:*

$$\mathrm{Vol}(M, \theta_Y)\Big|\min_M R_\theta\Big|^{-(n+1)} \leq \mathrm{Vol}(M, \theta) \leq \mathrm{Vol}(M, \theta_Y)\Big|\max_M R_\theta\Big|^{-(n+1)},$$

and each equality implies that R_θ is constant.

Corollary 1.6. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose CR Yamabe invariant satisfies $Y(M, \theta) < 0$. Then we have:*

$$\Big(\min_M R_\theta\Big)\mathrm{Vol}(M, \theta)^{\frac{1}{n+1}} \leq Y(M, \theta) \leq \Big(\max_M R_\theta\Big)\mathrm{Vol}(M, \theta)^{\frac{1}{n+1}},$$

and each equality implies that R_θ is constant.

The Riemannian version of Corollaries 1.5 and 1.6 was obtained in [7] and [5], respectively. See Corollary 16 in [7] and Lemma 1.6 in [5].

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We adopt the notation in [1].

Proof of Theorem 1.2. If $\tilde{\theta} = e^{2u}\theta$, then by the formula in p. 299 of [1] (see also [6]), their pseudo-Hermitian Ricci curvatures satisfy

$$\widetilde{R}_{\lambda\bar{\mu}} = R_{\lambda\bar{\mu}} - (n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) - \big(\Delta_\theta u + |\nabla_\theta u|^2_\theta\big)h_{\lambda\bar{\mu}}, \tag{2.1}$$

where $h_{\lambda\bar{\mu}}$ is the component of the Levi form (see p. 32 in [1]). Explicitly, let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T^{1,0}(M)$ on M, then

$$h_{\lambda\bar{\mu}} = L_\theta(T_\alpha, \overline{T_\mu})$$

where $L_\theta = -\sqrt{-1}\,d\theta$ is the Levi form with respect to θ. By assumption, $\mathrm{Ric}(\tilde{\theta}) = \mathrm{Ric}(\theta)$, (2.1) implies that

$$-(n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) - \big(\Delta_\theta u + |\nabla_\theta u|^2_\theta\big)h_{\lambda\bar{\mu}} = 0. \tag{2.2}$$

On the other hand, if we define the traceless Ricci tensor

$$B_{\lambda\bar{\mu}} = R_{\lambda\bar{\mu}} - \frac{R}{n}h_{\lambda\bar{\mu}}$$

where $R = R_{\lambda\bar{\mu}}h^{\lambda\bar{\mu}}$ is the Webster scalar curvature, then we have: