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Partial differential equations

A convergence result for the periodic unfolding method related to fast diffusion on manifolds

Un résultat de convergence pour la méthode d'éclatement périodique lié à la diffusion rapide sur des variétés

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ABSTRACT

Based on the periodic unfolding method in periodic homogenization, we deduce a convergence result for gradients of functions defined on connected, smooth, and periodic manifolds. Under the assumption of certain a-priori estimates of the gradient, which are typical for fast diffusion, the sum of a term involving a gradient with respect to the slow variable and one with respect to the fast variable is obtained in the homogenization limit. In addition, we show in a brief example how to apply this result and find for a reaction–diffusion equation defined on a periodic manifold that the homogenized equation contains a term describing macroscopic diffusion.

RÉSUMÉ

À l'aide de la méthode d'éclatement périodique, nous démontrons un résultat de convergence des gradients de fonctions définies sur des variétés connexes, différentiables et périodiques. Sous certaines conditions d'estimation du gradient, typiques de la diffusion rapide, nous obtenons à la limite d'homogénéisation la somme d'un gradient de la variable globale et d'un gradient de la variable locale. Un exemple illustre l'utilisation de ce résultat : pour une équation de réaction et diffusion définie sur une variété périodique, nous démontrons que l'équation homogénéisée contient un terme décrivant une diffusion globale.

1. Setting

The periodic unfolding method is a technique to homogenize partial differential equations. The main idea is the introduction of an operator $\mathcal{T}_\varepsilon$, which maps a function φ_ε defined on a finely structured periodic domain $\Omega_\varepsilon \subset \mathbb{R}^n$ to a function

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$\mathcal{T}_\varepsilon(\varphi_\varepsilon)$ defined on $\Omega \times Y$, where $Y = [0,1]^n$ is the periodicity cell. With $\Omega \subset \mathbb{R}^n$ being homogeneous, the domain of the function $\mathcal{T}_\varepsilon(\varphi_\varepsilon)$ is independent of ε and hence, we are able to use well-known convergence results from functional analysis.

The periodic unfolding method was developed in [3–6] based on ideas of [2]. It is the purpose of this note to extend these results by a weak compactness result for H^1-functions defined on a periodic manifold satisfying certain bounds (Theorem 4 below). These arise in problems involving fast surface diffusion, cf. Section 4. For utilization in the proof of Theorem 4, we also show an extension lemma (Lemma 5), which may be useful in related contexts as well.

We briefly describe the setting and summarize important results required in what follows. Let $\Omega \subset \mathbb{R}^n$ be a domain, and further let $\Omega_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k+Y) \cap \Omega$ and $\Gamma_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} \varepsilon(k+\Gamma) \cap \Omega$ be sets with periodic fine-structure with unit cell $Y = [0,1]^n$ and a smooth manifold $\Gamma \subset Y$, such that Γ_ε is smooth and connected and Ω is representable by a finite union of axis-parallel cuboids, each of which is assumed to have corner coordinates in $\mathbb{Q}^n$. This last technical assumption is required in order to use a certain extension operator, cf. Remark 6. Note that there also exist recent works in the context of periodic unfolding and manifolds, where the manifold itself is not periodic but has a periodic pattern on its surface [7], which is different from the setting considered here.

Let $\Xi_\varepsilon := \{\xi \in \mathbb{Z}^n \mid \varepsilon(\xi+Y) \subset \Omega\}$ and $\hat{\Omega}_\varepsilon := \text{interior}\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi+\overline{Y})\}$.

For every $z \in \mathbb{R}^n$, we define $[z]_Y$ as the unique integer combination $\sum_{i=1}^n k_i e_i$ of the periods such that $\{z\}_Y = z - [z]_Y \in Y$. The periodic unfolding operator $\mathcal{T}_\varepsilon$ is then defined as follows [4]:

Definition 1. Let $\varphi \in L^p(\Omega_\varepsilon)$, $p \in [1,\infty]$. For any $\varepsilon > 0$, we define $\mathcal{T}_\varepsilon : L^p(\Omega_\varepsilon) \to L^p(\Omega \times Y)$ such that

$$\big[\mathcal{T}_\varepsilon(\varphi)\big](x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \quad \text{a.e. for } (x,y) \in \hat{\Omega}_\varepsilon \times Y, \qquad \big[\mathcal{T}_\varepsilon(\varphi)\big](x,y) = 0 \quad \text{a.e. for } (x,y) \in \Omega\backslash\hat{\Omega}_\varepsilon \times Y.$$

The main advantage of using the periodic unfolding operator is that $\mathcal{T}_\varepsilon(\varphi)$ is defined on the fixed domain $\Omega \times Y$ even for varying ε. Thus, we may use standard convergence results from functional analysis. For example, the following weak compactness result in H^1 is proven in [5]. It is the main ingredient in identifying the limit problem when homogenizing typical reaction–diffusion equations stated on Ω_ε.

Theorem 2. *For every $\varepsilon > 0$, let φ_ε be in $H^1(\Omega_\varepsilon)$ and let $\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ be bounded independently of ε. Then there exist $\varphi \in H^1(\Omega)$ and $\hat{\varphi} \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ such that, up to a subsequence,*

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} \varphi \quad \text{weakly in } L^2\big(\Omega, H^1_{\mathrm{per}}(Y)\big), \qquad \mathcal{T}_\varepsilon(\nabla_x\varphi_\varepsilon) \overset{\varepsilon\to 0}{\rightharpoonup} \nabla_x\varphi + \nabla_y\hat{\varphi} \quad \text{weakly in } L^2\big(\Omega, L^2(Y)\big).$$

When internal boundary terms are to be homogenized, e.g. arising from interface conditions or surface concentrations, the boundary periodic unfolding operator $\mathcal{T}^b_\varepsilon$ is introduced. It is defined as follows, see [6].

Definition 3. Let $\varphi \in L^p(\Gamma_\varepsilon)$, $p \in [1,\infty]$. Then the boundary periodic unfolding operator $\mathcal{T}^b_\varepsilon : L^p(\Gamma_\varepsilon) \to L^p(\Omega \times \Gamma)$ is defined as

$$\mathcal{T}^b_\varepsilon(\varphi)(x,y) = \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{a.e. for } (x,y) \in \hat{\Omega}_\varepsilon \times \Gamma, \qquad \mathcal{T}^b_\varepsilon(\varphi)(x,y) = 0 \quad \text{a.e. for } (x,y) \in \Omega\backslash\hat{\Omega}_\varepsilon \times \Gamma.$$

It is well known in periodic homogenization that different scalings with the homogenization parameter lead to different limit behavior (see e.g. [14], where weak compactness results in the spirit of Theorem 2 are discussed for different scalings). The canonical scaling of surface terms is ε, that of surface gradients is ε^3, which is due to the fact that $|\Gamma_\varepsilon| \sim \varepsilon^{-1}$ in the limit. For these scalings, associated with slow diffusion, local (or microscopic) diffusion in the unit cell, i.e. with respect to the y-variable, is obtained in the homogenization limit [1,13].

The purpose of this contribution is to extend the results to fast diffusion, associated with a scaling of the surface gradients with ε^1. It turns out that this leads to global (or macroscopic) diffusion, i.e. with respect to the x-variable, in the homogenization limit.

In what follows, we formulate the main result in Section 2, present the proof in Section 3 and apply it to homogenize a prototypical diffusion problem in Section 4.

2. Statement of the main result

The main result is the following weak compactness result for H^1-functions defined on a manifold Γ_ε.

Theorem 4. *Let $\varphi_\varepsilon \in H^1(\Gamma_\varepsilon)$ be a sequence of functions with*

$$\varepsilon\|\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\nabla_\Gamma\varphi_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \leqslant C,$$