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Partial differential equations/Numerical analysis

Optimal decay rates for the stabilization of a string network

Taux de décroissance optimaux pour la stabilisation d'un réseau de cordes

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A R T I C L E I N F O

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A B S T R A C T

We study the decay of the energy for a degenerate network of strings, and obtain optimal decay rates when the lengths are all equal. We also define a classical space semi-discretization and compare the results with the exact method introduced by Ammari and Jellouli (Appl. Math. 52 (4) (2007) 327–343; Bull. Belg. Math. Soc. Simon Stevin 4 (2010) 717–735).

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R É S U M É

On étudie la décroissance de l'énergie pour un réseau de cordes dégénéré, et on obtient des taux de décroissance optimaux lorsque les longueurs sont égales. On définit aussi un semi-discrétisation classique et on compare les résultats avec ceux de la méthode exacte introduite par Ammari et Jellouli (Appl. Math. 52 (4) (2007) 327–343 ; Bull. Belg. Math. Soc. Simon Stevin 4 (2010) 717–735).

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1. Introduction

We recall that the dissipation condition $\partial_x u(t,0) = \alpha \partial_t u(t,0)$, at the origin of a vibrating elastic string fixed at its end point, stabilizes this string. More precisely, if $E(t)$ denotes the energy of the solution of the wave equation $\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0$ on $(0,\ell)$ subject to the initial condition $u(0,.) = a$ and $\partial_t u(0,.) = b$, then $E(t) \le Ce^{-\gamma_\alpha t}$ where $\gamma_\alpha = \frac{1}{\ell}\log|\frac{1+\alpha}{1-\alpha}|$. Moreover, we remark that if $\alpha = 1$, then $E(t) = 0$, $\forall t \ge 2\ell$. Thus, the value 1 is the best choice of α that makes the system is an equilibrium state. The situation is completely different in the case when we consider a network of strings.

The problem of stabilization of nondegenerate network of strings was studied by K. Ammari, M. Jellouli and M. Khenissi in [5], [1] and [2]. When the strings are coupled at a common end in a star-shaped configuration, it is proven in [1] that the solutions are not exponentially stable in the energy space, in the nondegenerate case. In the particular case of two strings, which is equivalent to the pointwise stabilization of one string, such a problem has been studied in [4]; exponential stabilization is obtained if and only if the lengths satisfy $\frac{\ell_1}{\ell_1+\ell_2} = \frac{p}{q}$, with p and q odd numbers, and the best decay rate, when fixing the total length $\ell_1+\ell_2$, is obtained when the lengths are equal ($\ell_1 = \ell_2 = \ell$), with $\gamma = \frac{\ln(3)}{\ell}$ as the best decay

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rate. Such a problem has been also considered in [7], with different boundary conditions; in the degenerate case, the energy limit was identified and it was proven that the solution decays exponentially toward that limit.

We consider here the case of a *degenerate* network of vibrating elastic strings when the pointwise feedback acts in the root of the tree (tree-shaped network). Note that in the *nondegenerate* case, it is proven in [5] that the solutions are not exponentially stable in the energy space. We calculate the limit energy $E_\infty := \lim_{t\to+\infty} E(t)$ and we show that the decrease from $E(t)$ to E_∞ is exponential, giving the best decay rate, when the lengths are equal. Finally, we give numerical results in Section 3, which confirm the theoretical results.

2. E_∞ and best decay rate

Let $N \geq 3$. We consider the initial data $((a_j)_{1\leq j\leq N}, (b_j)_{1\leq j\leq N}) \in \mathcal{H} := \prod_{j=1}^N H^2(0,\ell_j) \times \prod_{j=1}^N H^1(0,\ell_j)$, satisfying the compatibility conditions

$$a_1'(0) = \alpha b_1(0), \quad a_1'(\ell_1) = \sum_{j=2}^N a_j'(0) \quad \text{and} \quad a_j(\ell_j) = 0, \quad a_1(\ell_1) = a_j(0), \ j = 2, \ldots, N, \tag{1}$$

and the system of partial differential equations

$$(S): \begin{cases} \partial_t^2 u_j(t,x) - \partial_x^2 u_j(t,x) = 0, & t > 0, \ x \in (0,\ell_j), \ j = 1, \ldots, N, \\ u_j(t,0) = u_1(t,\ell_1) \quad \text{and} \quad u_j(t,\ell_j) = 0, & t \geq 0, \ j = 2, \ldots, N, \\ \partial_x u_1(t,0) = \alpha \partial_t u_1(t,0), & t \geq 0 \ (\alpha > 0), \\ \partial_x u_1(t,\ell_1) = \displaystyle\sum_{j=2}^N \partial_x u_j(t,0), & t \geq 0, \\ u_j(0,x) = a_j(x), \qquad \partial_t u_j(0,x) = b_j(x), & x \in [0,\ell_j], \ j = 1, \ldots, N. \end{cases}$$

We define the energy of the solution of (S) by

$$E(t) = \frac{1}{2} \sum_{j=1}^N \|\partial_t u_j(t)\|_{L^2(0,\ell_j)}^2 + \frac{1}{2} \sum_{j=1}^N \|\partial_x u_j(t)\|_{L^2(0,\ell_j)}^2 \tag{2}$$

and we prove the following theorems in the case where $\ell_j = \ell$, $j = 1, \ldots, N$ (we denote by $\| \ \|$ the norm of $L^2(0,\ell)$):

Theorem 2.1. *If $((a_j)_{1\leq j\leq N}, (b_j)_{1\leq j\leq N}) \in \mathcal{H}$ verify (1), then the energy limit is given by*

$$E_\infty = \frac{1}{2(N-1)} \sum_{j=2}^N \sum_{k=j+1}^N \left(\|(a_k' - a_j')\|^2 + \|(b_k - b_j)\|^2 \right). \tag{3}$$

Now, we denote by $\alpha_0 = \frac{2\sqrt{N-1}}{N}$, $\Delta = \frac{4(\alpha^2N^2 - 4N + 4)}{(\alpha+1)^2}$ and $\lambda = -\frac{\frac{2(N-2)}{\alpha+1} + \sqrt{\Delta}}{2N}$.[1]

Theorem 2.2. *There is a constant $C = C_N(\alpha) > 0$ such that for any initial data $((a_j)_{1\leq j\leq N}, (b_j)_{1\leq j\leq N}) \in \mathcal{H}$ satisfying (1) the following inequality holds:*

$$0 \leq E(t) - E_\infty \leq C\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^N a_j' \right\|^2 + \left\| \sum_{j=2}^N b_j \right\|^2 \right) \frac{e^{-\gamma t}}{|\Delta|}, \quad \text{if } \alpha \neq \alpha_0, \tag{4}$$

$$0 \leq E(t) - E_\infty \leq C\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^N a_j' \right\|^2 + \left\| \sum_{j=2}^N b_j \right\|^2 \right) t^2 e^{-\gamma_0 t}, \quad \text{if } \alpha = \alpha_0, \tag{5}$$

where $\gamma = \frac{1}{\ell} \log \frac{1}{|\lambda|} > 0$ and $\gamma_0 = \frac{1}{\ell} \log \frac{N+2\sqrt{N-1}}{N-2}$. The best decay rate is achieved in the sense that for all $\alpha > 0$, there exists initial data such that (4) and (5) become an equivalence:

$$E(t) - E_\infty \geq C e^{-\gamma t}, \quad \text{or,} \quad \text{for } \alpha = \alpha_0, \quad E(t) - E_\infty \geq C t^2 e^{-\gamma_0 t}, \quad \text{with } C > 0,$$

and $\forall \alpha \neq \alpha_0$, we have $\gamma < \gamma_0$.

Remark 1. The case $N = 2$ (with ℓ_1, ℓ_2 not necessarily of same size) can be recasted into the case $N = 1$ (with $\ell = \ell_1 + \ell_2$).

[1] $\sqrt{\Delta} = i\sqrt{-\Delta}$ if $\Delta < 0$. Note that, $\Delta < 0$ (resp. $\Delta = 0$) if and only if $\alpha \in]0, \alpha_0[$ (resp. $\alpha = \alpha_0$).

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