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Partial differential equations

# Averaged control and observation of parameter-depending wave equations

## *Contrôle et observation en moyenne d'équations des ondes dépendant de paramètres*

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ABSTRACT

We analyze the problem of averaged observability and control of wave equations.
This topic is motivated by the control of parameter-dependent systems of wave equations. We look for controls ensuring the controllability of the averages of the states with respect to the parameter. This turns out to be equivalent to the problem of averaged observation in which one aims at recovering the energy of the initial data of the adjoint system by measurements done on its averages, under the assumption that the initial data of all the components of the adjoint system coincide.
The problem under consideration is weaker than the classical notion of simultaneous observation and control.
The method of proof uses propagation arguments based on H-measures or microlocal defect measures that reduce the problem to non-standard unique-continuation issues.
Using transmutation techniques, we also derive some results on the averaged observation and control of parameter-dependent heat equations.



RÉSUMÉ

On étudie le problème de l'observation et contrôle en moyenne d'équations des ondes.
Ce sujet est motivé par le contrôle d'équations des ondes dépendant de paramètres. On s'intéresse à la contrôlabilité des moyennes des états par rapport aux paramètres. Ceci équivaut au problème de l'observation des états adjoints dépendant des paramètres, mais tous avec les mêmes données initiales, en utilisant l'observation des moyennes.
Le problème considéré est plus faible que celui de la contrôlabilité ou de l'observabilité simultanées étudié antérieurement.
La méthode de démonstration s'appuie sur des arguments de propagation qui utilisent les H-mesures ou mesures de défaut microlocales, qui réduisent le problème à des questions de continuation unique.

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En utilisant des arguments de transmutation, on obtient aussi quelques résultats pour le contrôle et l'observation en moyenne pour des équations paraboliques dépendant de paramètres.


## 1. Introduction

We analyze the problem of controlling the averaged value of a system of parameter-dependent wave equations by a single control. The problem is relevant in applications in which the control has to be chosen independently of the parameter value, in a robust manner.

This notion was previously introduced in [12] both in the context of finite-dimensional linear and PDE systems.

The problem is equivalent to that of averaged observability, in which we try to determine the energy of an initial datum for the parameter-dependent wave equations, by means of simply observing their averages with respect to the parameter.

These notions are weaker than those of simultaneous control and observation ([6] and [1]). In simultaneous control, all the wave equations, regardless of the value of the parameter, need to be controlled by the same control. In simultaneous observation, the initial data of the solutions whose average is observed are supposed to depend arbitrarily on the parameter and not all to be the same as when dealing with averaged observation.

Our main results of averaged observation and, by duality, of averaged control, employ tools of microlocal analysis, and, more precisely, propagation arguments based on the use of microlocal defect measures or H-measures introduced independently by P. Gérard [9] and L. Tartar [11].[1] We refer the reader to the mentioned articles for the properties of these measures (localization, propagation, etc.) used in this Note. Our methods are strongly inspired by those developed in earlier works for the observation and control of hyperbolic equations [3,4] and [6], among others.

In the next section, we discuss the simplest case of a system of two distinct wave equations. Results on simultaneous observability, together with the appropriate assumptions and relations to the existing results are given in Section 3. Using transmutation techniques, the results obtained for wave equations are then transferred to systems of heat equations (Section 4). We close the paper by pointing towards some open problems and future directions of research.

## 2. Averaged observability

As mentioned above, we consider the case where the system under consideration only involves two modes, depending on the velocity of propagation of solutions, denoted respectively by $c_1$, $c_2$ and $u_1$, $u_2$:

$$\begin{aligned}
&\partial_{tt} u_i - \operatorname{div}\big(c_i(\mathbf{x})\nabla u_i\big) = 0, \quad (t, \mathbf{x}) \in \mathbf{R}^+ \times \Omega \\
&u_i(0, \cdot) = u^0 \in \mathrm{L}^2(\Omega) \\
&\partial_t u_i(0, \cdot) = \tilde{u}^0 \in \mathrm{H}^{-1}(\Omega), \quad i = 1, 2,
\end{aligned} \tag{1}$$

where the space domain $\Omega$ is assumed to be a compact manifold without boundary. As for the coefficients entering the system, we assume that they are bounded from below by a positive constant. Furthermore, if not stated otherwise, it is assumed that $c_1$ is of class $\mathrm{C}^{1,1}$, thus ensuring the well-posedness of the bicharacteristic flow of the corresponding wave operator, while $c_2$ is merely continuous.

We investigate the conditions under which one can recover the energy of the initial data, which are the same for both components, by observing the average of solutions, $\theta u_1 + (1-\theta)u_2$, with a parameter $\theta \in \langle 0, 1]$.

The main result of this Note is as follows:

**Theorem 2.1.** *Suppose the equations' coefficients satisfy*

$$c_1(\mathbf{x}) - c_2(\mathbf{x}) \neq 0, \quad \mathbf{x} \in \omega, \tag{2}$$

*where $\omega$ is an open subset of $\Omega$ and $T$ is a time such that $\langle 0, T\rangle \times \omega$ satisfies the Geometric Control Condition (GCC, [5]) for the first equation.*

*Then, for any $\theta \in \langle 0, 1]$ there exists a constant $C_\theta$ such that the following estimate holds*

$$E(0) := \|u^0\|^2_{\mathrm{L}^2} + \|\tilde{u}^0\|^2_{\mathrm{H}^{-1}} \leq C_\theta \int_0^T \int_\omega \big|\theta u_1 + (1-\theta)u_2\big|^2 \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{3}$$

[1] In the sequel, for simplicity, we shall use the terminology of H-measures.