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Numerical analysis

A simple nonconforming quadrilateral finite element

Un élément fini non conforme quadrilatéral simple

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ABSTRACT

We introduce and analyze a simple nonconforming quadrilateral finite element and then we derive optimal a priori error estimates for arbitrary regular quadrilaterals. The idea of extension to some non-conforming elements for three-dimensional hexahedrons is also presented.

RÉSUMÉ

Dans ce travail, nous présentons et analysons un élément fini non conforme en quadrangles. Nous obtenons une estimation d'erreur a priori optimale pour des quadrangles réguliers arbitraires. Nous présentons également l'idée d'extension tridimensionnelle de cet élément.

Version française abrégée

Les éléments finis non conformes jouissent de bonnes propriétés de stabilité ; ils ont intéressé de nombreux mathématiciens – voir par exemple [2,3,5,9,10]. En 1973, Crouzeix et Raviart [4] ont été les premiers à examiner l'élément triangulaire linéaire non conforme avec trois degrés de liberté situés au milieu des arêtes. Cet élément non conforme P_1–P_0, le plus simple, a été utilisé avec succès pour résoudre les équations de Stokes stationnaires. Ensuite, Han (1984) a proposé un élément rectangulaire non conforme pour résoudre les équations de Stokes stationnaires [7] et de Navier–Stokes [6]. Pour les équations de Stokes, Rannacher et Turek [11] ont introduit en 1992 un élément fini Q_1-non conforme sur les quadrilatères convexes.

Dans cette note, nous présentons et analysons un élément fini quadrilatère simple non conforme, pour lequel on donne une condition nécessaire et suffisante (2.4) pour l'unisolvance, et établissons des estimations d'erreur a priori optimales pour des quadrilatères réguliers arbitraires. Des choix particuliers des fonctions B_k définies plus loin nous permettent de retrouver des éléments finis classiques. Nous présentons également l'idée d'obtention des résultats analogues pour certains éléments non conformes en trois dimensions pour les hexaèdres.

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1. Introduction

Crouzeix and Raviart [4] (1973) first considered the nonconforming linear *triangular* element with three nodes located at midpoints of edges. This simplest nonconforming P_1–P_0 element (piecewise linear elements for velocity and piecewise constants for pressure), which is called C–R triangular element today, has been successfully utilized for solving the stationary Stokes equations. Han (1984) proposed a nonconforming *rectangular* element for solving the stationary Stokes equations [6] and Navier–Stokes equations [7]. For the Stokes equations, Rannacher and Turek [11] (1992) introduced the so-called 'rotated' Q_1-nonconforming finite element on arbitrarily convex quadrilaterals. The corresponding local finite element spaces are obtained by rotating the mixed term of the bilinear element, and the local degrees of freedom are either the average of the function over the edge or its value at the midpoint of the edge. Nonconforming finite elements enjoy better stability properties compared to the conforming finite elements [8]. They have attracted the attention of many mathematicians, see, e.g. [2,3,5,9,10,12].

In this note, we introduce and analyze a simple nonconforming quadrilateral finite element and derive optimal a priori error estimates for arbitrary regular quadrilaterals. The idea of extension to some non-conforming elements for three-dimensional hexahedrons is also presented.

2. A quadrilateral nonconforming element

Let K be a convex and non-degenerate quadrilateral domain with vertices $\{a_i\}_{1\le i\le 4}$ numbered counterclockwise. Denote by $\{e_i\}_{1\le i\le 4}$ its edges $[a_i, a_{i+1}]$, in which the indices are numbered modulo four. In order to introduce our nonconforming finite element, let us assume that we are given a function $B_K \in \mathcal{C}^0(K)$ and define the two sets:

$$R_K = P_1(K) + \operatorname{span}\{B_K\}, \qquad \Sigma_K := \left\{\mu : v \to \int_{e_i} v\,\mathrm{d}\sigma,\ i = 1, \ldots, 4\right\}.$$

In the following we use the functional

$$l_K(f) := \sum_{i=1}^{4} \frac{(-1)^i}{\operatorname{meas}(e_i)} \int_{e_i} f\,\mathrm{d}\sigma. \tag{2.1}$$

It is easy to show that the functional l_K defines a linear functional on $\mathcal{C}^0(K)$, and satisfies

$$l_K(p) = 0, \quad p \in P_1(K). \tag{2.2}$$

It is also worth noting that when q belongs to R_K, then $q = p_q + \alpha_q B_K$, with $p_q \in P_1(K)$ and $\alpha_q \in \mathbb{R}$; so, using (2.2), we get for all $q \in R_K$:

$$l_K(q) := \alpha_q l_K(B_k). \tag{2.3}$$

The next characterization result is the starting point of our nonconforming finite element.

Theorem 1. *Let l be the functional given by (2.3). Then the triple (K, R_K, Σ_K) is a finite element if and only if*

$$l_K(B_K) \neq 0. \tag{2.4}$$

Proof. *Necessity:* Let us assume to the contrary that $l_K(B_K) = 0$. Then, using the fact $l_K(p) = 0$, for all p affine function, we deduce from (2.3) that $l_k(q) = 0$, for all $q \in R_K$. Thus, there are no functions $q_j \in R_K$ satisfying $\int_{e_i} q_j \mathrm{d}\sigma = \delta_i^j$, $i = 1, \ldots, 4$. Consequently, the triple (K, R_K, Σ_K) is not a finite element. Hence, the necessary condition (2.4) is satisfied.

Sufficiency: Let us assume that condition (2.4) is satisfied. Then, using (2.2), we deduce that $B_K \notin P_1(K)$ and then $\dim(R_K) = 4 = \operatorname{Card}(\Sigma_K)$. Let $q \in R_K$ such that

$$\int_{e_i} q\,\mathrm{d}\sigma = 0, \quad i = 1, \ldots, 4. \tag{2.5}$$

Now using (2.2), it is easy to see that $0 = l_K(q) = \alpha l_K(B_K)$, then $\alpha = 0$, and hence $q \in P_1(K)$. Thus, by using (2.5), the fact that normal derivatives of affine functions are constant and Green's formula we get:

$$\int_K |\nabla q|^2\,\mathrm{d}x = -\int_K q\Delta q\,\mathrm{d}x + \sum_{i=1}^{4} \int_{e_i} q \frac{\partial q}{\partial \nu_i}\,\mathrm{d}\sigma = \sum_{i=1}^{4} \frac{\partial q}{\partial \nu_i} \int_{e_i} q\,\mathrm{d}\sigma = 0.$$