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Lie algebras

On the structure and arithmeticity of lattice envelopes ☆

Sur la structure et l'arithméticité des groupes enveloppant un réseau

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ABSTRACT

We announce results about the structure and arithmeticity of all possible lattice embeddings of a class of countable groups that encompasses all linear groups with simple Zariski closure, all groups with non-vanishing first ℓ^2-Betti number, non-elementary acylindrically hyperbolic groups, and non-elementary convergence groups.

RÉSUMÉ

Nous nous intéressons à l'ensemble des plongements possibles d'un groupe dénombrable comme réseau dans un groupe localement compact. Pour une grande classe de groupes dénombrables, nous annonçons des résultats de structure et d'arithméticité de tels plongements. Cette classe contient tous les groupes linéaires dont l'adhérence de Zariski est simple, les groupes dont le premier nombre de Betti ℓ^2 est non nul, les groupes hyperboliques acylindriques et les groupes de convergence.

1. Introduction

Let Γ be a countable group. We are concerned with the study of its lattice envelopes, i.e. the locally compact groups containing Γ as a lattice. We aim at structural results that impose no restrictions on the ambient locally compact group and only abstract group-theoretic conditions on Γ. We say that Γ satisfies (†) if every finite index subgroup of a quotient of Γ by a finite normal subgroup

(†1) is not isomorphic to a product of two infinite groups, and
(†2) does not possess infinite amenable commensurated subgroups, and
(†3) satisfies: for a normal subgroup N and a commensurated subgroup M with $N \cap M = \{1\}$, there exists a finite index subgroup $M' < M$ such that N and M' commute.

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The relevance of (†1) should be clear, the relevance of (†2) is that it yields an information about all possible lattice envelopes of Γ [1]:

Proposition 1.1. *Let Γ be a lattice in a locally compact group G. If Γ has no infinite amenable commensurated subgroups, then the amenable radical $R(G)$ of G is compact.*

The role of (†3) is less transparent, but be aware of the obvious observation: if M and N are both normal with $N \cap M = \{1\}$, then M and N commute. There are lattices Γ in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{Aut}(T)$, where T is the universal cover of the 1-skeleton $B^{(1)}$ of the Bruhat–Tits building of $\mathrm{SL}_n(\mathbb{Q}_p)$ (see [4, 6.C] and [3, Prop. 1.8]). They are built in such a way that $N := \pi_1(B^{(1)})$, which is a free group of infinite rank, is a normal subgroup of Γ. Let $U < \mathrm{Aut}(T)$ be the stabilizer of a vertex. Then $M := \Gamma \cap (\mathrm{SL}_n(\mathbb{R}) \times U)$ is commensurated, but $M, N < \Gamma$ violate (†3). Moreover, this group Γ satisfies (†1) and (†2).

Proposition 1.2. *Linear groups with semi-simple Zariski closure satisfy conditions (†2) and (†3). Groups with some positive ℓ^2-Betti number satisfy condition (†2). All the (†) conditions are satisfied by all linear groups with simple Zariski closure, by all groups with positive first ℓ^2-Betti number, and by all non-elementary acylindrically hyperbolic groups and convergence groups.*

For a concise formulation of our main result, we introduce the following notion of S-arithmetic lattice embeddings up to tree extension: let K be a number field. Let $\mathbf{H}$ be a connected, absolutely simple adjoint K-group, and let S be a set of places of K that contains every infinite place for which $\mathbf{H}$ is isotropic and at least one finite place for which $\mathbf{H}$ is isotropic. Let $\mathcal{O}_S \subset K$ denote the S-integers. The (diagonal) inclusion of $\mathbf{H}(\mathcal{O}_S)$ into $\prod_{v\in S} \mathbf{H}(K_v)$ is the prototype example of an S-arithmetic lattice. Let H be a group obtained from $\prod_{v\in S} \mathbf{H}(K_v)^+$ by possibly replacing each factor $\mathbf{H}(K_v)^+$ with K_v-rank 1 by an intermediate closed subgroup $\mathbf{H}(K_v)^+ < D < \mathrm{Aut}(T)$ where T is the Bruhat–Tits tree of $\mathbf{H}(K_v)$. The lattice embedding of $\mathbf{H}(\mathcal{O}_S) \cap H$ into H is called an *S-arithmetic lattice embedding up to tree extension*.

A typical example is $\mathrm{SL}_2(\mathbb{Z}[1/p])$ embedded diagonally as a lattice into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$. The latter is a closed cocompact subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{Aut}(T_{p+1})$, where T_{p+1} is the Bruhat–Tits tree of $\mathrm{SL}_2(\mathbb{Q}_p)$, i.e. a $(p+1)$-regular tree. So $\mathrm{SL}_2(\mathbb{Z}[1/p]) < \mathrm{SL}_2(\mathbb{R}) \times \mathrm{Aut}(T_{p+1})$ is an S-arithmetic lattice embedding up to tree extension. We now state the main result [1]:

Theorem 1.3. *Let Γ be a finitely generated group satisfying (†), e.g. one of the groups considered in Proposition 1.2. Then every embedding of Γ as a lattice into a locally compact group G is, up to passage to finite index subgroups and dividing out a normal compact subgroup of G, isomorphic to one of the following cases:*

(i) *an irreducible lattice in a center-free, semi-simple Lie group without compact factors;*
(ii) *an S-arithmetic lattice embedding up to tree extension, where S is a finite set of places;*
(iii) *a lattice in a totally disconnected group with trivial amenable radical.*

The same conclusion holds true if one replaces the assumption that Γ is finitely generated with the assumption that G is compactly generated.

Finite generation of Γ implies compact generation of any locally compact group containing Γ as a lattice. The examples above for $n \geq 3$ show that condition (†3) in Theorem 1.3 is indispensable. Since non-uniform lattices with a uniform upper bound on the order of finite subgroups do not exist in totally disconnected groups, our main theorem yields the following classification of non-uniform lattice embeddings.

Corollary 1.4. *Let Γ be a group that satisfies (†) and admits a uniform upper bound on the order of all finite subgroups. Then every non-uniform lattice embedding of Γ into a compactly generated locally compact group G is, up to passage to finite index subgroups and dividing out a normal compact subgroup of G, either a lattice in a center-free, semi-simple Lie group without compact factors or an S-arithmetic lattice embedding up to tree extension.*

For groups Γ that are not lattices in Lie groups (classical or S-arithmetic) but satisfy (†) and are torsion-free, possible lattice envelopes are uniform and totally disconnected (Theorem 1.3 (iii)). This can be used, for example, to show that Gromov–Thurston groups $\Gamma = \pi_1(M)$, where M is a compact manifold that admits an almost hyperbolic Riemannian metric, but not a hyperbolic one, have only the trivial lattice embeddings $\Gamma < \Gamma \ltimes K$, where K is a compact group.

The following arithmeticity theorem [1] is at the core of the proof of Theorem 1.3. Actually, it is a more general version that is used, in which we drop condition (†1) (see the comment in Step 3 of Section 2).

In the proof of Theorem 1.3, we only need Theorem 1.5 in the case where D, thus $L \times D$, is compactly generated, which means that the set S of primes is finite. Caprace–Monod [4, Theorem 5.20] show Theorem 1.5 for compactly generated D and under the hypothesis that L is the k-points of a simple k-group (where k is a local field), but the latter hypothesis is too restrictive for our purposes. Moreover, our proof of Theorem 1.5 does not become much easier if we assume compact generation from the beginning. Regardless of its role in Theorem 1.3, we consider the following result as a first step in the classification of lattices in locally compact groups that are not necessarily compactly generated.