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Dirac families for loop groups as matrix factorizations

Familles d'opérateurs de Dirac pour les groupes de lacets et factorisations en matrices

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ABSTRACT

We identify the category of integrable lowest-weight representations of the loop group LG of a compact Lie group G with the category of twisted, conjugation-equivariant *curved Fredholm complexes* on the group G: namely, the twisted, equivariant *matrix factorizations* of a super-potential built from the loop rotation action on LG. This lifts the isomorphism of K-groups of [3–5] to an equivalence of categories. The construction uses families of Dirac operators.

RÉSUMÉ

On identifie la catégorie des représentations intégrables de plus bas poids du groupe de lacets LG d'un groupe de Lie compact G avec la catégorie des complexes de Fredholm tordus, courbés et équivariants pour conjugaison sur le groupe G : plus précisément, les *factorisations en matrices* d'un potentiel provenant de la rotation des lacets dans LG. Cette construction relève l'isomorphisme de K-groupes de [3–5] en une équivalence de catégories. La construction fait appel aux familles d'opérateurs de Dirac.

1. Introduction and background

The group LG of smooth loops in a compact Lie group G has a remarkable class of linear representations whose structure parallels the theory for compact Lie groups [10]. The defining stipulation is the existence of a circle action on the representation, with finite-dimensional eigenspaces and spectrum bounded below, intertwining with the loop rotation action on LG. We denote the rotation circle by $\mathbb{T}_r$; its infinitesimal generator L_0 represents the *energy* in a conformal field theory.

Noteworthy is the *projective nature* of these representations, described (when G is semi-simple) by a *level* $h \in H^3_G(G;\mathbb{Z})$ in the equivariant cohomology for the adjoint action of G on itself. The representation category $\mathfrak{Rep}^h(LG)$ at a given level h is semi-simple, with finitely many simple isomorphism classes. Irreducibles are classified by their *lowest weight* (plus some supplementary data when G is not simply connected [5, Ch. IV]).

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In a series of papers [3–5], the authors, jointly with Michael Hopkins, construct $K^0\mathfrak{Rep}^h(LG)$ in terms of a twisted, conjugation-equivariant topological K-theory group. To wit, when G is connected, as we shall assume throughout this paper,[1] we have

$$K^0\,\mathfrak{Rep}^h(LG) \cong K_G^{\tau+\dim G}(G), \tag{1.1}$$

with a twisting $\tau \in H_G^3(G;\mathbb{Z})$ related to h, as explained below.

Remark 1.1. One loop group novelty is a *braided tensor* structure[2] on $\mathfrak{Rep}^h(LG)$. The structure arises from the *fusion product* of representations, relevant to 2-dimensional conformal field theory. The K-group in (1.1) carries a Pontryagin product, and the multiplications match in (1.1).

The map from representations to topological K-classes is implemented by the following *Dirac family*. Calling $\mathcal{A}$ the space of connections on the trivial G-bundle over S^1, the quotient stack $[G:G]$ under conjugation is equivalent to $[\mathcal{A}:LG]$ under the gauge action, via the holonomy map $\mathcal{A} \to G$. Denote by $\mathbf{S}^{\pm}$ the (lowest-weight) modules of spinors for the loop space $L\mathfrak{g}$ of the Lie algebra and by $\psi(A):\mathbf{S}^{\pm} \to \mathbf{S}^{\mp}$ the action of a Clifford generator A, for $d + Adt \in \mathcal{A}$. A representation $\mathbf{H}$ of LG leads to a family of Fredholm operators over $\mathcal{A}$,

$$\not{D}_A : \mathbf{H}\otimes\mathbf{S}^+ \to \mathbf{H}\otimes\mathbf{S}^-, \quad \not{D}_A := \not{D}_0 + \mathrm{i}\psi(A) \tag{1.2}$$

where $\not{D}_0$ is built from a certain Dirac operator [7] on the loop group.[3] The family is projectively LG-equivariant; dividing out by the subgroup $\Omega G \subset LG$ of based loops leads to a projective, G-equivariant Fredholm complex on G, whose K-theory class $\left[(\not{D}_\bullet, \mathbf{H}\otimes\mathbf{S}^{\pm})\right] \in K_G^{\tau+*}(G)$ is the image of $\mathbf{H}$ in the isomorphism (1.1). When $\dim G$ is odd, $\mathbf{S}^+ = \mathbf{S}^-$ and skew-adjointness of $\not{D}_A$ leads instead to a class in K^1. The twisting τ is the level of $\mathbf{H}\otimes\mathbf{S}$ as an LG-representation, with a (G-dependent) shift from the level h of $\mathbf{H}$.

The shifts are best explained in the world of super-categories, with $\mathbb{Z}/2$ gradings on morphisms and objects; odd simple objects have as endomorphisms the rank one Clifford algebra Cliff(1), and in the semi-simple case, they contribute a free generator to K^1 instead of K^0. Consider the τ-projective representations of LG with compatible action of Cliff($L\mathfrak{g}$), thinking of them as modules for the (not so well-defined) crossed product $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$. They form a semi-simple super-category $\mathfrak{SRep}^\tau$, and the isomorphism (1.1) becomes

$$K^*\,\mathfrak{SRep}^\tau\,(LG \ltimes \mathrm{Cliff}(L\mathfrak{g})) \cong K_G^{\tau+*}(G) \tag{1.3}$$

with the advantage of having no shift in degree or twisting. (For simply connected G, both sides live in degree $* = \dim\mathfrak{g}$, but both parities can be present for general G.) This isomorphism is induced by the Dirac families of (1.2): a super-representation $\mathbf{SH}^{\pm}$ of $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$ can be coupled to the Dirac operators $\not{D}_A$ without a choice of factorization as $\mathbf{H}\otimes\mathbf{S}^{\pm}$.

2. The main result

There is a curious mismatch in (1.3): the isomorphism is induced by a functor of underlying Abelian categories, from $\mathbb{Z}/2$-graded representations to twisted Fredholm bundles over G, but this functor is far from an equivalence. The category $\mathfrak{SRep}^\tau$ is semi-simple (in the graded sense discussed), but that of twisted Fredholm complexes is not so; we can even produce continua of non-isomorphic objects in any given K-class, by compact perturbation of a Fredholm family.

Here, we redress this problem by incorporating a *super-potential*, a celebrity in the algebraic geometry of 2-dimensional physics (the "B-model"). As explained by Orlov[4] [8], this deforms the category of complexes of vector bundles into that of *matrix factorizations*: the 2-*periodic, curved complexes* with curvature equal to the super-potential W. Our W has Morse critical points, leading to a semi-simple super-category with one generator for each critical point; the generators are precisely the Dirac families of (1.2) on irreducible LG-representations. The artifice of introducing W is redeemed by its natural topological origin in the *loop rotation* $\mathbb{T}_r$-action on the stack $[G:G]$. The $\mathbb{T}_r$-action is evident in the presentation $[\mathcal{A}:LG]$, but it rigidifies to a $B\mathbb{Z}$-action on the stack. Furthermore, for twistings τ transgressed from BG, the $B\mathbb{Z}$-action lifts to the G-equivariant *gerbe* G^τ over G which underlies the K-theory twisting. The logarithm of this lift is $2\pi\mathrm{i}W$.

Remark 2.1. The conceptual description of a super-potential as logarithm of a $B\mathbb{Z}$-action on a category of sheaves is worked out in [9]; the matrix factorization category is the *Tate fixed-point category* for the $B\mathbb{Z}$-action. For varieties, W is a function and $\exp(2\pi\mathrm{i}W)$ generates a $B\mathbb{Z}$-action on sheaves; on a stack, a geometric underlying action can also be present, as in this case. With respect to [9], our W_τ below should be re-scaled to take integer values at all critical points; we will omit this detail in order to better connect with the formulas in [4,5].

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[1] Twisted loop groups show up when G is disconnected [5].

[2] When G is not simply connected, there is a constraint on h.

[3] The normalized operator $(-2)^{-1/2}\not{D}_0$ is the square root G_0 of L_0 in the super-Virasoro algebra.

[4] Orlov discusses complex algebraic vector bundles; we found no exposition for equivariant Fredholm complexes in topology, and a discussion is planned for our follow-up paper.