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Semicontinuous viscosity solutions for quasiconvex Hamiltonians ☆

Solutions de viscosité semicontinues des hamiltoniens quasi-convexes

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ABSTRACT

The main theorem connecting convex Hamiltonians and semicontinuous viscosity solutions due to Barron and Jensen is extended to quasiconvex Hamiltonians. Some applications are indicated.

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RÉSUMÉ

Le théorème principal reliant les hamiltoniens convexes et les solutions de viscosité semicontinues, due à Barron et Jensen, est étendu aux hamiltoniens quasi-convexes. Quelques applications sont indiquées.

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1. Introduction

In 1990, an extension of the Crandall–Lions notion of viscosity solutions for first order Hamilton–Jacobi equations was presented in [4]. This extension showed that when the Hamiltonian is convex in the gradient variable, touching the function from below and testing to see if the result gives zero, i.e., $p \in D^- u(x) \Longrightarrow H(x, u(x), p) = 0$ is equivalent to the Crandall–Lions definition. This was important because, for lower semicontinuous functions, we may only be able to touch from below, and many problems only have semicontinuous solutions. The comparison principle for semicontinuous solutions was established in [4]. In 1993, H. Frankowska developed in [8] a nonsmooth approach to the uniqueness of semicontinuous viscosity solutions.

The aim of this note is to show that the idea of the connection between convex Hamiltonians and semicontinuous solutions extends to quasiconvex (equivalently, level convex) Hamiltonians, a much broader class of functions. Essentially all of the convex Hamiltonian results may be extended to quasiconvex Hamiltonians, except for one significant class, namely equations of the form $u_t + H(t, x, u, Du) = 0$. Assuming such an equation is quasiconvex in (u_t, Du) will force H to be convex in Du. Thus, finite-horizon Bolza or Lagrange problems are not covered. On the other hand, equations of the form $\lambda u + H(x, u, Du) = 0$ and time-dependent equations arising in optimal stopping and L^∞ control and calculus of variations problems are covered. Some applications for L^∞ problems are presented, as well as a representation result for a viscosity solution of $H(x, u, Du) = 0$.

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2. Semicontinuous solutions for quasiconvex Hamiltonians

The problem we consider is:

$$H\big(x, u(x), Du(x)\big) = 0, \quad x \in \Omega \subset \mathbb{R}^n. \tag{2.1}$$

In this paper, we will focus on Hamiltonians that are quasiconvex (also called level convex) in p:

$$H\big(x, r, \lambda p + (1-\lambda)q\big) \leqslant \max\big\{H(x, r, p), H(x, r, q)\big\}, \quad 0 \leqslant \lambda \leqslant 1. \tag{2.2}$$

Equivalently, $E_\alpha(x, r) = \{p \in \mathbb{R}^n \mid H(x, r, p) \leqslant \alpha\}$ is convex for all $\alpha \in \mathbb{R}$. One of the fundamental features of quasiconvex functions is the extended Jensen inequality.

Proposition 2.1. *(See [4,6].) Let $\mathcal{A} \subset \mathbb{R}^n$ be convex and $f : \mathcal{A} \to \mathbb{R}$ be quasiconvex and μ be a probability measure on $\mathcal{A}$. Then*:

$$f\left(\int \varphi(x)\,\mathrm{d}\mu(x)\right) \leqslant \mu - \operatorname*{ess\,sup}_{x \in \mathcal{A}} f\big(\varphi(x)\big) \tag{2.3}$$

for any $\varphi \in L^1(\mu)$.

Proof. Consider $E_\alpha = \{z \in \mathcal{A} \mid f(z) \leqslant \alpha = \mu - \operatorname{ess\,sup}_{x \in \mathcal{A}} f(\varphi(x))\}$. Since f is quasiconvex, E_α is convex and $\varphi(x) \in E_\alpha$ for μ − almost all $x \in \mathcal{A}$. But then $\int \varphi(x)\,\mathrm{d}\mu(x) \in E_\alpha$. □

Definition 2.2 *(Crandall–Lions).* A bounded function u is a viscosity subsolution of (2.1) if $p \in D^+u^{usc}(x)$ implies $H_{lsc}(x, u, p) \leqslant 0$, and is a viscosity supersolution of (2.1) if $p \in D^-u_{lsc}(x)$ implies $H^{usc}(x, u_{lsc}, p) \geqslant 0$. Here a subscript *lsc* refers to the lower semicontinuous envelope of the function, and a superscript *usc* denotes the upper semicontinuous envelope.

Our goal in this section is to prove the following theorem. This extends one of the main theorems in [4] to quasiconvex Hamiltonians.

Theorem 2.3. *Let $H(x, r, p)$ be continuous and satisfy $p \mapsto H(x, r, p)$ is quasiconvex and*

$$\big|H(x, r, p) - H(y, s, p)\big| \leqslant \omega\big(|x - y|\big(|p| + 1\big) + |r - s|\big), \quad x \in \Omega \subset \mathbb{R}^n, r \in \mathbb{R}, p \in \mathbb{R}^n, \tag{2.4}$$

for $\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\omega(0) = 0$. Then a uniformly continuous function $u : \Omega \to \mathbb{R}$ is a Crandall–Lions solution of $H(x, u(x), Du(x)) = 0$, $x \in \Omega$, if and only if

$$x \in \Omega, \ p \in D^-u(x) \quad \Longrightarrow \quad H\big(x, u(x), p\big) = 0. \tag{2.5}$$

For brevity and in view of the literature since the introduction of such semicontinuous solutions in [4], we will refer to u as a BJ solution if it satisfies the condition (2.5).

There are two steps to prove this theorem. The first one assumes the stronger condition that u is Lipschitz.

Proposition 2.4. *Let $u \in Lip(\Omega)$ and assume $p \mapsto H(x, r, p)$ is quasiconvex. Assume (see Ishii [9]) the condition*:

$$H \text{ is continuous and } \forall R > 0, \quad \limsup_{\varepsilon \to 0}\big\{\big|H(y, s, p) - H(x, r, p)\big| \big| |y - x| < \varepsilon, \ |r - s| < \varepsilon, \ |p| \leqslant R\big\} = 0. \tag{2.6}$$

Then u is a Crandall–Lions viscosity subsolution of $H(x, u, Du) \leqslant 0$ if and only if u satisfies the condition $p \in D^-u(x) \Longrightarrow H(x, u(x), p) \leqslant 0$, i.e., u is a BJ subsolution.

Proof. We use Ishii's [9] improved formulation of Theorem 1.1 in [4] that if u is bounded and lower semicontinuous (respectively, upper semicontinuous), then:

$$D^+u(x) \subset \bigcap_{r>0} \overline{co}\left(\bigcup_{|x-y|<r} D^-u(y)\right), \quad \text{respectively,} \quad D^-u(x) \subset \bigcap_{r>0} \overline{co}\left(\bigcup_{|x-y|<r} D^+u(y)\right). \tag{2.7}$$

Assume u is a Crandall–Lions subsolution. By (2.7), if $p \in D^-u(x)$, there is sequence $p_k \to p$, and $\{x_i\}$, $\{\lambda_i\}$, $\{q_i\}$, $i = 1, 2, \ldots, N(k)$, $\exists N(k) \in \mathbb{Z}^+$, such that $|x - x_i| < \frac{1}{k}$, $q_i \in D^+u(x_i)$, $0 \leqslant \lambda_i \leqslant 1$, and $p_k = \sum \lambda_i q_i$, $\sum \lambda_i = 1$. Thus $H(x_i, u(x_i), q_i) \leqslant 0$. Observe that $|q_i| \leqslant Lip(u)$. Then, using (2.6), one shows $H(x, u(x), q_i) \leqslant o(\frac{1}{k}) \to 0$ as $k \to \infty$, $1 \leqslant i \leqslant N$. By quasiconvexity,

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