C. R. Acad. Sci. Paris, Ser. I 351 (2013) 743–746

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

A new relation between the condensation index of complex sequences and the null controllability of parabolic systems

Une nouvelle relation entre l'indice de condensation des suites complexes et la contrôlabilité à zéro des systèmes paraboliques

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ARTICLE INFO

Article history:
Received 25 June 2013
Accepted after revision 3 September 2013
Available online 21 October 2013

Presented by the Editorial Board

ABSTRACT

In this note, we present a new result that relates the condensation index of a sequence of complex numbers with the null controllability of parabolic systems. We show that a minimal time is required for controllability. The results are used to prove the boundary controllability of some coupled parabolic equations.

RÉSUMÉ

On annonce un résultat qui relie l'indice de condensation des suites complexes et la contrôlabilité à zéro des systèmes paraboliques. On montre qu'un temps minimal de contrôle est nécessaire. Ces résultats sont ensuite utilisés pour étudier la contrôlabilité à zéro par le bord de quelques systèmes paraboliques.

1. Notation and main results

Let $\mathbb{X}$ be a Hilbert space on $\mathbb{C}$ with norm and inner product respectively denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Let us consider $\{\phi_k\}_{k\geqslant 1}$ a Riesz basis of $\mathbb{X}$ and denote $\{\psi_k\}_{k\geqslant 1}$ the corresponding biorthogonal sequence to $\{\phi_k\}_{k\geqslant 1}$. Also consider a sequence $\Lambda=\{\lambda_k\}_{k\geqslant 1}\subset\mathbb{C}$, with $\lambda_i\neq\lambda_k$ for all $i\neq k$, satisfying for a $\delta>0$,

$$\Re(\lambda_k)\geqslant\delta|\lambda_k|>0,\quad \forall k\geqslant 1,\ \text{and}\ \sum_{k\geqslant 1}\frac{1}{|\lambda_k|}<\infty. \tag{1}$$

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.09.014

Denote by $\mathbb{X}_{-1}$ the completion of $\mathbb{X}$ with respect to the norm: $\|y\|_{-1} := (\sum_{k\geqslant 1} \frac{|(y,\psi_k)|^2}{|\lambda_k|^2})^{1/2}$. Also the Hilbert space $(\mathbb{X}_1, \|\cdot\|_1)$ is defined by $\mathbb{X}_1 := \{y \in \mathbb{X}: \|y\|_1 < \infty\}$ with $\|y\|_1^2 = \sum_{k\geqslant 1} |\lambda_k|^2 |(y,\psi_k)|^2$. Furthermore, let $\mathcal{A} : \mathcal{D}(\mathcal{A}) = \mathbb{X}_1 \subset \mathbb{X} \to \mathbb{X}$ be the operator given by:

$$\mathcal{A} = -\sum_{k\geqslant 1} \lambda_k (\cdot, \psi_k)\phi_k. \tag{2}$$

Let us fix $T > 0$ a real number and $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ (so $\mathcal{B}^* \in \mathcal{L}((\mathbb{X}_{-1})', \mathbb{C}) \equiv \mathbb{X}_{-1}$). We consider:

$$y' = \mathcal{A}y + \mathcal{B}u \quad \text{on } (0,T); \qquad y(0) = y_0 \in \mathbb{X}. \tag{3}$$

In System (3), $u \in L^2(0,T;\mathbb{C})$ is the control that acts on the system by means of the operator $\mathcal{B}$. We assume that $\mathcal{B}$ is an admissible control operator for the semigroup generated by $\mathcal{A}$, i.e., for a positive time T^* one has $\mathrm{R}(L_{T^*}) \subset \mathbb{X}$, where $L_T u = \int_0^T e^{(T-s)\mathcal{A}} \mathcal{B}u(s)\,ds$. System (3) is approximately controllable in $\mathbb{X}$ at time $T > 0$ if for every $y_0 \in \mathbb{X}$, $\mathcal{R}(T) = \{y(T) = e^{T\mathcal{A}} y_0 + L_T u$ with $u \in L^2(0,T;\mathbb{C})\}$ is dense in $\mathbb{X}$ and System (3) is *null controllable* in $\mathbb{X}$ at time $T > 0$ if for all $y_0 \in \mathbb{X}$, $0 \in \mathcal{R}(T)$. It is well known that the controllability properties of System (3) amount to appropriate properties of the so-called *adjoint system* to System (3). This adjoint system has the form:

$$-\varphi' = \mathcal{A}^*\varphi \quad \text{on } (0,T); \qquad \varphi(T) = \varphi_0 \in \mathbb{X}. \tag{4}$$

Observe that, for any $\varphi_0 \in \mathbb{X}$, System (4) admits a unique weak solution $\varphi \in C^0([0,T];\mathbb{X})$. Classical results (see e.g. [6, Theorem 11.2.1]) imply:

Theorem 1.1. *Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ generated by $\mathcal{A}$, with $\mathcal{A}$ given by (2), and $\Lambda = \{\lambda_k\}_{k\geqslant 1}$ is a complex sequence satisfying (1). Then, System (3) is approximately controllable in $\mathbb{X}$ at time T if and only if:*

$$b_k := \mathcal{B}^*\psi_k \neq 0, \quad \forall k \geqslant 1. \tag{5}$$

Moreover, (3) is null controllable in $\mathbb{X}$ at time T if and only if there exists a constant $C_T > 0$ such that:

$$\sum_{k\geqslant 1} e^{-2T\Re(\lambda_k)} |a_k|^2 \leqslant C_T \int_0^T \left| \sum_{k\geqslant 1} \bar{b}_k e^{-\lambda_k(T-t)} a_k \right|^2, \quad \forall \{a_k\}_{k\geqslant 1} \in \ell^2(\mathbb{C}). \tag{6}$$

Our main result reads as follows:

Theorem 1.2. *Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ and $\Lambda = \{\lambda_k\}_{k\geqslant 1}$ is a complex sequence satisfying respectively (5) and (1). For $z \in \mathbb{C}$, let us introduce $E(z) = \prod_{k=1}^{\infty}(1 - \frac{z^2}{\lambda_k^2})$ and $T_0 = \limsup(\frac{\log \frac{1}{|b_k|}}{\Re(\lambda_k)} + \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)})$. Then System (3) is null controllable for $T > T_0$ and is not null controllable for $T < T_0$.*

The *condensation index* of a sequence $\Lambda = \{\lambda_k\}_{k\geqslant 1} \subset \mathbb{C}$ satisfying (1) is the real number $c(\Lambda) = \limsup \frac{\log \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)}$, where the function E is given in Theorem 1.2. The condensation index is related to the overconvergence of Dirichlet series (see [5]). Observe that when $\lim \frac{\log |b_k|}{\Re(\lambda_k)} = 0$, then, $T_0 = c(\Lambda)$.

2. Idea of the proof of Theorem 1.2

The proof is technical and long and the details are given in [2]. For the positive result, we transform the control problem into a problem of moments. So we need to study the existence of biorthogonal families to complex exponentials and study some properties of these families. We have the following result:

Theorem 2.1. *Let $\Lambda = \{\lambda_k\}_{k\geqslant 1} \subset \mathbb{C}$ be a sequence satisfying (1) and fix $T \in (0,\infty]$. Let $A(\Lambda,T) = \overline{\mathrm{span}\{e^{-\lambda_k t}\colon k \geqslant 1\}}^{L^2(0,T;\mathbb{C})}$. Then, there exists a biorthogonal family $\{q_k\}_{k\geqslant 1} \subset A(\Lambda,T)$ to $\{e^{-\lambda_k t}\}_{k\geqslant 1}$ such that for any $\varepsilon > 0$ one has:*

$$C_{1,\varepsilon} \frac{e^{-\varepsilon\Re(\lambda_k)}}{|E'(\lambda_k)|} \leqslant \|q_k\|_{L^2(0,T;\mathbb{C})} \leqslant C_{2,\varepsilon} \frac{e^{\varepsilon\Re(\lambda_k)}}{|E'(\lambda_k)|}, \quad \forall k \geqslant 1, \tag{7}$$

where E is the function given in Theorem 1.2 and $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ are constants only depending on ε, Λ and T.