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Partial differential equations

Remarks on a lemma by Jacques-Louis Lions

Remarques sur un lemme de Jacques-Louis Lions

Chérif Amrouche [a], Philippe G. Ciarlet [b], Cristinel Mardare [c]

[a] *Laboratoire de mathématiques et de leurs applications, UMR–CNRS 5142, Université de Pau et des pays de l'Adour, avenue de l'Université, 64000 Pau, France*
[b] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[c] *Laboratoire Jacques-Louis-Lions, UMR CNRS 7598, Sorbonne Universités, Université Pierre-et-Marie-Curie, 4, place Jussieu, 75005 Paris, France*

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ABSTRACT

Let Ω be a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary $\partial\Omega$, the set Ω being locally on one side of $\partial\Omega$. It is shown in this Note that a fundamental characterization of the space $L^2(\Omega)$ due to Jacques-Louis Lions is in effect equivalent to a variety of other properties. One of the keys for establishing these equivalences is a specific "approximation lemma", itself one of these equivalent properties.

RÉSUMÉ

Soit Ω un ouvert borné et connexe de $\mathbb{R}^N$ de frontière $\partial\Omega$ lipschitzienne, l'ensemble Ω étant localement du même côté de $\partial\Omega$. On montre dans cette Note qu'une caractérisation fondamentale de l'espace $L^2(\Omega)$ due à Jacques-Louis Lions est en fait équivalente à un certain nombre d'autres propriétés. L'une des clés pour établir ces équivalences est un « lemme d'approximation » spécifique, qui constitue lui-même l'une de ces propriétés équivalentes.

1. Definitions and notations

In what follows, N designates a fixed integer ≥ 2. Unless otherwise specified, Latin indices range in the set $\{1, 2, \ldots, N\}$.

The notation V' designates the dual space of a topological vector space V and ${}_{V'}\langle\cdot,\cdot\rangle_V$ designates the duality between V' and V. Given a subspace W of a normed vector space V,

$$W^0 := \{v' \in V';\ {}_{V'}\langle v', w\rangle_V = 0 \text{ for all } w \in W\}$$

designates the polar set of W; if V is a Hilbert space, $W^\perp$ designates the orthogonal complement of W.

Let Ω be an open subset of $\mathbb{R}^N$ and let $x = (x_i)$ be a generic point in Ω. Partial derivative operators of the first order, in the classical sense or in the sense of distributions, are denoted $\partial_i := \partial/\partial x_i$. The space of functions that are indefinitely

E-mail addresses: cherif.amrouche@univ-pau.fr (C. Amrouche), mapgc@cityu.edu.hk (P.G. Ciarlet), mardare@ann.jussieu.fr (C. Mardare).

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differentiable in Ω and have compact supports in Ω is denoted $\mathcal{D}(\Omega)$ and the space of distributions on Ω is denoted $\mathcal{D}'(\Omega)$. If $f \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we also use the shorter notation $f(\varphi) := {}_{\mathcal{D}'(\Omega)}\langle f, \varphi\rangle_{\mathcal{D}(\Omega)}$. The notations $H^1(\Omega)$ and $H_0^1(\Omega)$ designate the usual Sobolev spaces, and the notation $H^{-1}(\Omega)$ designates the dual space of $H_0^1(\Omega)$ endowed with the norm of $H^1(\Omega)$. Finally, we define the space

$$L_0^2(\Omega) := \left\{ f \in L^2(\Omega); \int_\Omega f \, \mathrm{d}x = 0 \right\}.$$

Spaces of functions and vector fields defined over Ω are respectively denoted by italic capitals and boldface Roman capitals.

The *gradient operator* $\mathbf{grad} : \mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}}'(\Omega)$ is defined for each $f \in \mathcal{D}'(\Omega)$ by

$${}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{grad}\, f, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} := -{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{\varphi}\rangle_{\mathcal{D}(\Omega)} \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega).$$

Note that, when restricted to $L^2(\Omega)$, the mapping $\mathbf{grad} : L^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ satisfies

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \mathbf{grad}\, f, \boldsymbol{v}\rangle_{\boldsymbol{H}_0^1(\Omega)} = -\int_\Omega f \operatorname{div} \boldsymbol{v} \, \mathrm{d}x \quad \text{for all } f \in L^2(\Omega) \text{ and all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega).$$

This shows that the operator $\mathbf{grad} : L_0^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is the dual operator of $-\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$. It also easily implies that, if the open set Ω is connected, the operator $\mathbf{grad} : L_0^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ is one-to-one.

The *curl operator* $\mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \to \mathcal{D}(\Omega; \mathbb{R}^{N(N-1)/2})$ is defined for each $\boldsymbol{h} = (h_i) \in \boldsymbol{\mathcal{D}}'(\Omega)$ by

$$(\mathbf{curl}\, \boldsymbol{h})_{ij} = \partial_i h_j - \partial_j h_i \quad \text{for each } i < j.$$

A *domain* Ω in $\mathbb{R}^N$ is a bounded and connected open subset Ω of $\mathbb{R}^N$ whose boundary $\partial\Omega$ is Lipschitz-continuous, the set Ω being locally on the same side of $\partial\Omega$.

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and, given $r > 0$, let $B(x; r) := \{y \in \mathbb{R}^N;\ |y - x| < r\}$. An open subset of $\mathbb{R}^N$ is *starlike with respect to an open ball* $B(x; r)$ if, for each $z \in \Omega$, the convex hull of the set $\{z\} \cup B(x; r)$ is contained in the set Ω.

2. Jacques-Louis Lions' lemma

Let Ω be a domain in $\mathbb{R}^N$. The *classical J.-L. Lions lemma* asserts that $f \in H^{-1}(\Omega)$ and $\mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$ implies $f \in L^2(\Omega)$. Its first published proof, under the assumption that the boundary of Ω is smooth, appeared in Duvaut and Lions [7]; see also Tartar [12] for a different proof, under the same assumption. The first proof for a general domain is due to Geymonat and Suquet [8].

That the assumption $f \in H^{-1}(\Omega)$ can be replaced by the more general assumption $f \in \mathcal{D}'(\Omega)$ was established by Borchers and Sohr [4], as a consequence of a result of Bogovskii [3], who gave a constructive proof that the operator $\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ is onto; then by Amrouche and Girault [2], as a consequence of an inequality due to Nečas [11] (also used in [8]), asserting the existence of a constant $C_0(\Omega)$ such that

$$\|f\|_{L^2(\Omega)} \le C_0(\Omega)\big(\|f\|_{H^{-1}(\Omega)} + \|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}\big) \quad \text{for all } f \in L^2(\Omega).$$

Note that the results of both [2], [4] and [8] hold for a general domain Ω. We shall call *J.-L. Lions' lemma* this stronger result, which thus asserts that

$$f \in \mathcal{D}'(\Omega) \quad \text{and} \quad \mathbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \quad \text{implies } f \in L^2(\Omega).$$

Both the classical J.-L. Lions lemma and its more general version above are fundamental results from functional analysis, with many crucial applications to partial differential equations (see, e.g., Sections 6.14 to 6.19 in [5]).

3. Jacques-Louis Lions' lemma and its relation to other basic results

The main objective of this Note is to show, by means of a sequence of implications (Theorems 1 to 5), that the above properties, viz., both versions of J.-L. Lions' lemma, the surjectivity of div, and the inequality of Nečas, as well as other properties, are in fact *equivalent*. The key to establishing these equivalences is an *approximation lemma* (cf. Theorem 4), which constitutes in effect one of the equivalence properties and appears to be new. In so doing, we provide in addition what we believe are substantially simpler proofs of those implications than those that are already known in the literature. Detailed proofs will appear in [1].