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Complex analysis

The weighted log canonical threshold

Le seuil log-canonique pondéré

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ABSTRACT

In this note, we show how to apply the original L^2-extension theorem of Ohsawa and Takegoshi to the standard basis of a multiplier ideal sheaf associated with a plurisubharmonic function. In this way, we are able to reprove the strong openness conjecture and to obtain an effective version of the semicontinuity theorem for weighted log canonical thresholds.

RÉSUMÉ

Dans cet article, nous montrons comment appliquer la version originelle du théorème d'extension L^2 de Ohsawa et Takegoshi à la base standard d'un faisceau d'idéaux multiplicateurs associé à une fonction plurisousharmonique. Ceci nous permet de redémontrer la conjecture d'ouverture forte et d'obtenir une version effective du théorème de semi-continuité pour les seuils log-canoniques pondérés.

1. Introduction and main results

Let Ω be a domain in $\mathbb{C}^n$ and φ in the set $\mathrm{PSH}(\Omega)$ of plurisubharmonic functions on Ω. Following Demailly and Kollár [10], we introduce the log canonical threshold of φ at a point $z_0 \in \Omega$

$$c_\varphi(z_0) = \sup\{c > 0\colon\ \mathrm{e}^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\} \in (0, +\infty].$$

It is an invariant of the singularity of φ at z_0. We refer to [5,13,6,7,9,10,14,18,19,22,23,25,26] for further information about this number. In [10], Demailly and Kollár stated the following openness conjecture.

Conjecture. *The set* $\{c > 0\colon\ \mathrm{e}^{-2c\varphi}$ *is* L^1 *on a neighborhood of* $z_0\}$ *equals the open interval* $(0, c_\varphi(z_0))$.

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In 2005, this conjecture was proved in dimension 2 by Favre and Jonsson [12,20,21]. In 2013, Berndtsson [2] completely proved it in arbitrary dimension. For every holomorphic function f on Ω, we introduce the *weighted log canonical threshold* of φ with weight f at z_0:

$$c_{\varphi,f}(z_0) = \sup\{c > 0: \ |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\} \in (0, +\infty].$$

Recently, Guan and Zhou used a sophisticated version of the L^2-extension theorem of Ohsawa and Takegoshi in combination with the curve selection lemma, to prove the "strong" openness conjecture, i.e. the analogue openness statement for weighted thresholds $c_{\varphi,f}(z_0)$, and a related semi-continuity theorem for the weighted log canonical threshold [16,17]. In this note, we show how one can apply the original version [24] of the L^2-extension theorem to the members of a standard basis for a multiplier ideal sheaf of holomorphic functions associated with a plurisubharmonic function φ. In this way, by means of a simple induction on dimension, we reprove the strong openness conjecture, and give an effective version of the semicontinuity theorem for weighted log canonical thresholds. The main results are contained in the following theorem.

Main theorem. *Let f be a holomorphic function on an open set Ω in $\mathbb{C}^n$ and let $\varphi \in \mathrm{PSH}(\Omega)$.*

(i) ("Semicontinuity theorem") *Assume that $\int_{\Omega'} e^{-2c\varphi}\, dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$. Then for $\psi \in \mathrm{PSH}(\Omega')$, there exists $\delta = \delta(c, \varphi, \Omega', z_0) > 0$ such that $\|\psi - \varphi\|_{L^1(\Omega')} \leqslant \delta$ implies $c_\psi(z_0) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in L^1 on every relatively compact open subset $\Omega'' \Subset \Omega'$.*

(ii) ("Strong effective openness") *Assume that $\int_{\Omega'} |f|^2 e^{-2c\varphi}\, dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$. When $\psi \in \mathrm{PSH}(\Omega')$ converges to φ in $L^1(\Omega')$ with $\psi \leqslant \varphi$, the function $|f|^2 e^{-2c\psi}$ converges to $|f|^2 e^{-2c\varphi}$ in L^1 norm on every relatively compact open subset $\Omega'' \Subset \Omega'$.*

Corollary 1.1 *("Strong openness"). For any plurisubharmonic function φ on a neighborhood of a point $z_0 \in \mathbb{C}^n$, the set $\{c > 0: \ |f|^2 \times e^{-2c\varphi}$ is L^1 on a neighborhood of $z_0\}$ is an open interval $(0, c_{\varphi,f}(z_0))$.*

Corollary 1.2 *("Convergence from below"). If $\psi \leqslant \varphi$ converges to φ in a neighborhood of $z_0 \in \mathbb{C}^n$, then $c_{\psi,f}(z_0) \leqslant c_{\varphi,f}(z_0)$ converges to $c_{\varphi,f}(z_0)$.*

In fact, after subtracting a large constant from φ, we can assume $\varphi \leqslant 0$ in both corollaries. Then Corollary 1.1 is a consequence of assertion (ii) of the main theorem when we take Ω' small enough and $\psi = (1+\delta)\varphi$ with $\delta \searrow 0$. In Corollary 1.2, we have by definition $c_{\psi,f}(z_0) \leqslant c_{\varphi,f}(z_0)$ for $\psi \leqslant \varphi$, but again (ii) shows that $c_{\psi,f}(z_0)$ becomes $\geqslant c$ for any given value $c \in (0, c_{\varphi,f}(z_0))$, whenever $\|\psi - \varphi\|_{L^1(\Omega')}$ is sufficiently small.

Remark 1.3. One cannot remove condition $\psi \leqslant \varphi$ in assertion (ii) of the main theorem. Indeed, let us choose $f(z) = z_1$, $\varphi(z) = \log|z_1|$ and $\varphi_j(z) = \log|z_1 + \frac{z_2}{j}|$, for $j \geqslant 1$. We have $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$, however $c_{\varphi_j,f}(0) = 1 < c_{\varphi,f}(0) = 2$ for all $j \geqslant 1$. On the other hand, condition (i) does not require any given inequality between φ and ψ. Modulo Berndtsson's solution of the openness conjecture, (i) follows from the effective semicontinuity result of [10], but (like Guan and Zhou) we reprove here both by a direct and much easier method.

Remark 1.4. As in Guan and Zhou [17], one can reformulate Corollary 1.1 in terms of multiplier ideal sheaves. Denote by $\mathcal{I}(c\varphi)$ the sheaf of germs of holomorphic functions $f \in \mathcal{O}_{\mathbb{C}^n,z}$ such that $\int_U |f|^2 e^{-2c\varphi}\, dV_{2n} < +\infty$ on some neighborhood U of z (it is known by [23] that this is a coherent ideal sheaf over Ω, but we will not use this property here). Then at every point $z \in \Omega$ we have

$$\mathcal{I}(c\varphi) = \mathcal{I}_+(c\varphi) := \lim_{\epsilon \searrow 0} \mathcal{I}\big((1+\epsilon)c\varphi\big).$$

2. Proof of the main theorem

We equip the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions at 0 with the homogeneous lexicographic order of monomials $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, that is, $z_1^{\alpha_1} \dots z_n^{\alpha_n} < z_1^{\beta_1} \dots z_n^{\beta_n}$ if and only if $|\alpha| = \alpha_1 + \dots + \alpha_n < |\beta| = \beta_1 + \dots + \beta_n$ or $|\alpha| = |\beta|$ and $\alpha_i < \beta_i$ for the first index i with $\alpha_i \neq \beta_i$. For each $f(z) = a_{\alpha^1} z^{\alpha^1} + a_{\alpha^2} z^{\alpha^2} + \cdots$ with $a_{\alpha^j} \neq 0$, $j \geqslant 1$ and $z^{\alpha^1} < z^{\alpha^2} < \cdots$, we define the *initial coefficient*, *initial monomial* and *initial term* of f to be respectively $\mathrm{IC}(f) = a_{\alpha^1}$, $\mathrm{IM}(f) = z^{\alpha^1}$, $\mathrm{IT}(f) = a_{\alpha^1} z^{\alpha^1}$, and the support of f to be $\mathrm{SUPP}(f) = \{z^{\alpha^1}, z^{\alpha^2}, \dots\}$. For any ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbb{C}^n,0}$, we define $\mathrm{IM}(\mathcal{I})$ to be the ideal generated by $\{\mathrm{IM}(f)\}_{\{f \in \mathcal{I}\}}$. First, we recall the division theorem of Hironaka and the concept of standard basis of an ideal.

Division theorem of Hironaka. *(See [15,1,3,4,11].) Let $f, g_1, \dots, g_k \in \mathcal{O}_{\mathbb{C}^n,0}$. Then there exist $h_1, \dots, h_k, s \in \mathcal{O}_{\mathbb{C}^n,0}$ such that*

$$f = h_1 g_1 + \dots + h_k g_k + s,$$