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Differential geometry

A characterization of balanced manifolds

Une caractérisation des variétés semi-kählériennes

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ABSTRACT

A Hermitian metric on a complex manifold is Kähler if and only if it approximates the Euclidean metric to order 2 at each point, in a suitable coordinate system. We prove here an analogous characterization of balanced metrics, namely, a Hermitian metric is balanced if and only if its fundamental form ω has closed trace and $\omega_{i,j}(z)$ does not contain linear terms involving $z_i, z_j, \overline{z_i}, \overline{z_j}$, for each point, in a suitable coordinate system.

RÉSUMÉ

Une métrique hermitienne de forme fondamentale ω sur une variété complexe M est kählérienne si et seulement s'il existe un système de cordonnées z sur un voisinage de chaque point de M, tel que la composante linéaire de $\omega_{i,j}(z)$ s'annule. On montre ici un critère de semi-kählérianité, à savoir qu'une métrique hermitienne de forme ω sur M est semi-kählérienne si et seulement s'il existe un système de cordonnées z sur un voisinage de chaque point de M, tel que la part linéaire de $\omega_{i,j}(z)$ ne contienne pas $z_i, z_j, \overline{z_i}, \overline{z_j}$, et que la trace de ω soit fermée.

1. Introduction

The general context of this note is the wide collection of works originated by the paper of M.L. Michelson in 1983 [5], where balanced manifolds were introduced and studied; we also refer to our joint papers in collaboration with G. Bassanelli on this subject (see, e.g., [1,2]).

The point of view of Michelson is the following: "the condition of being balanced is, in a strong sense, dual to that of being Kähler", because she starts from a very general problem, namely, how to choose a good Hermitian metric on a complex manifold.

By contrast, our philosophy is to consider Kähler and balanced manifolds from the point of view of p-Kähler manifolds (see Section 2): Kähler manifolds correspond to the case $p = 1$, while balanced manifolds correspond to the case $p = n - 1$, where n is the dimension of the manifold.

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The approaches are quite different: in the first case, one studies Hermitian metrics, in the other one, only positive differential forms.

Starting from a Hermitian metric h, we can consider both the associated canonical Hermitian connection and the Kähler form of the metric. In the first case, we can look at the torsion tensor T_h of h, and at the torsion 1-form τ_h of h: h is Kähler if and only if $T_h = 0$, h is balanced if and only if $\tau_h = 0$.

Alternatively, if ω_h is the Kähler form of h, and d_h^* is the formal adjoint of d with respect to the metric h, one gets that h is Kähler if and only if $d\omega_h = 0$, h is balanced if and only if $d_h^* \omega_h = 0$, that is, if and only if $d(\omega_h)^{n-1} = 0$.

Notice that, at an elementary level, the notion of Kähler metric is introduced in a different way: since it is important, especially on compact manifolds, to ensure a link between Laplacians, one says that a Hermitian metric is Kähler if and only if it approximates the Euclidean metric to order 2 at each point [4, p. 106]. In this paper, we investigate to which extent balanced metrics are dual to Kähler metrics from this point of view, which we feel to be the easiest and the most basic one to look at. The case $n = 2$ allows us to compare our result with the classical one.

This note will pursue our past philosophy of considering $(n-1)$-Kähler forms (see Theorem 3.1). As a corollary (see Proposition 3.3) we give the characterization result of balanced metrics. The motivation is to stress that, also when we do not have a good Hermitian metric on the manifold, it is possible to handle and solve problems at the level of differential forms.

2. Notation and preliminary results

Let M be a complex manifold of dimension $n \geqslant 3$, let p be an integer, $1 \leqslant p \leqslant n-1$, and let $\sigma_p = i^{p^2} 2^{-p}$.

Definition 2.1. M is a p-Kähler manifold if it has a closed transverse (i.e. strictly weakly positive) (p,p)-form Ω, which is called a p-Kähler form.

For $p = 1$, a transverse form is the fundamental form of a Hermitian metric, so that a 1-Kähler manifold is simply a Kähler manifold, and we can look at 1-Kähler *metrics*.

The case $p = n-1$ was studied by Michelson in [5], where $(n-1)$-Kähler manifolds are called *balanced* manifolds.

For $p = n-1$, we get a Hermitian metric too, because every transverse $(n-1,n-1)$-form Ω is in fact given by $\Omega = \omega^{n-1}$, where ω is a transverse $(1,1)$-form (the proof uses a comparison between the eigenvalues of Ω_x and those of ω_x, see [5, p. 279]); we say that ω is associated with a balanced metric.

When $n = 2$, a balanced metric (manifold) is simply a Kähler metric (manifold).

3. Characterization of balanced manifolds

Theorem 3.1. *Let M be a complex n-dimensional manifold, and let Ω be a real $(n-1,n-1)$-form on M. Then Ω is an $(n-1)$-Kähler form if and only if, for every $p \in M$, there is a holomorphic coordinate system $(w_1, \ldots, w_n)$ centered at p such that*

$$\Omega = \sigma_{n-1} \sum_{i,j=1}^{n} \Omega_{i,\bar{j}} \widehat{dw_i} \wedge \widehat{d\overline{w}_j},$$

with

(i) $\Omega_{i,\bar{j}}(0) = \delta_{i,j}$,
(ii) $\Omega_{i,\bar{j}}(w)$ *does not contain linear terms involving* $w_i, w_j, \overline{w_i}, \overline{w_j}$,
(iii) $\mathrm{d}(\mathrm{tr}\, \Omega_{i,\bar{j}})(0) = 0$.

Proof. One part of the proof is easy: in fact, when Ω satisfies (i) for every $p \in M$, then $\Omega|_p = \sigma_{n-1} \sum_{i=1}^{n} \widehat{dw_i} \wedge \widehat{d\overline{w_i}} > 0$. Moreover,

$$\partial \Omega|_p = 0 \quad \Longleftrightarrow \quad \forall j, \quad \sum_{i=1}^{n} (-1)^{i-1} (\partial_i \Omega_{i,\bar{j}})(0) = 0. \tag{1}$$

By condition (ii), $\Omega_{i,\bar{j}}$ does not contain linear terms involving w_i, so that $\partial \Omega|_p = 0$, and thus $d\Omega|_p = 0$, that is, Ω is closed.

Let now suppose that Ω is an $(n-1)$-Kähler form, let $p \in M$, and let $(z_1, \ldots, z_n)$ be a generic holomorphic coordinate system centered at p (that is, $z_j(p) = 0$). Here Ω is given by $\Omega = \sigma_{n-1} \sum_{i,j=1}^{n} \Omega_{i,\bar{j}} \widehat{dz_i} \wedge \widehat{d\overline{z_j}}$; since $\Omega > 0$, we can choose $(z_1, \ldots, z_n)$ such that $\Omega|_p$ is diagonalized, that is, $\Omega_{i,\bar{j}}(0) = \delta_{i,j}$. Thus