C. R. Acad. Sci. Paris, Ser. I 352 (2014) 431–434

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional analysis/Probability theory

Restricted isometry property for random matrices with heavy-tailed columns

Propriété d'isométrie restreinte de matrices aléatoires dont les colonnes sont à queues lourdes

Olivier Guédon [a], Alexander E. Litvak [b], Alain Pajor [a], Nicole Tomczak-Jaegermann [b,1]

[a] *Université Paris-Est, Laboratoire d'analyse et de mathématiques appliquées (UMR 8050), UPEMLV, 77454 Marne-la-Vallée, France*
[b] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada*

ARTICLE INFO

Article history:
Received 23 January 2014
Accepted 6 March 2014
Available online 27 March 2014

Presented by Gilles Pisier

ABSTRACT

Let A be a matrix whose columns $X_1, \dots, X_N$ are independent random vectors in $\mathbb{R}^n$. Assume that p-th moments of $\langle X_i, a\rangle$, $a \in S^{n-1}$, $i \leqslant N$, are uniformly bounded. For $p > 4$, we prove that with high probability A has the Restricted Isometry Property (RIP) provided that Euclidean norms $|X_i|$ are concentrated around $\sqrt{n}$ and that the covariance matrix is well approximated by the empirical covariance matrix provided that $\max_i |X_i| \leqslant C(nN)^{1/4}$. We also provide estimates for RIP when $\mathbb{E}\phi(|\langle X_i, a\rangle|) \leqslant 1$ for $\phi(t) = (1/2)\exp(t^\alpha)$, with $\alpha \in (0, 2]$.

RÉSUMÉ

Soit A une matrice dont les colonnes $X_1, \dots, X_N$ sont des vecteurs indépendants de $\mathbb{R}^n$. On suppose que les moments d'ordre p des $\langle X_i, a\rangle$, $a \in S^{n-1}$, $1 \leqslant i \leqslant N$ sont uniformément bornés pour $p > 4$. On démontre que si les normes euclidiennes des $|X_i|$ se concentrent autour de $\sqrt{n}$, la matrice A vérifie une propriété d'isométrie restreinte avec grande probabilité et que si $\max_i |X_i| \leqslant C(nN)^{1/4}$, la matrice de covariance empirique est une bonne approximation de la matrice de covariance. On démontre aussi une propriété d'isométrie restreinte quand $\mathbb{E}\phi(|\langle X_i, a\rangle|) \leqslant 1$ pour tout $a \in S^{n-1}$, $1 \leqslant i \leqslant N$ avec $\phi(t) = (1/2)\exp(t^\alpha)$ et $\alpha \in (0, 2]$.

E-mail addresses: olivier.guedon@u-pem.fr (O. Guédon), aelitvak@gmail.com (A.E. Litvak), alain.pajor@u-pem.fr (A. Pajor), nicole.tomczak@ualberta.ca (N. Tomczak-Jaegermann).

[1] This author holds the Canada Research Chair in Geometric Analysis.

http://dx.doi.org/10.1016/j.crma.2014.03.005
1631-073X/

1. Introduction

Our two main results go in two parallel directions: the Restricted Isometry Property abbreviated as RIP and a question of Kannan–Lovász–Simonovits about an approximation of a covariance matrix by empirical covariance matrices referred to below as KLS problem.

In this note, $X_1, \dots, X_N$ denote independent random vectors in $\mathbb{R}^n$ satisfying for some function ϕ

$$\forall 1 \leqslant i \leqslant N \quad \forall a \in S^{n-1} \quad \mathbb{E}\phi\big(\big|\langle X_i, a\rangle\big|\big) \leqslant 1. \tag{1}$$

We will focus on two choices of the function ϕ: $\phi(t) = t^p$, with $p > 4$, or $\phi(t) = (1/2)\exp(t^\alpha)$, with $\alpha \in (0, 2]$. The $n \times N$ matrix whose columns are $X_1, \dots, X_N$ will be denoted by A. As usual, C, C_1, ..., c, c_1, ... will always denote absolute positive constants, whose values may change from line to line.

2. Restricted Isometry Property (RIP)

We first recall the definition of the RIP, which was introduced in [6], in order to study the exact reconstruction problem by ℓ_1 minimization. It is noteworthy that the problem of reconstruction can be reformulated in terms of convex geometry, namely in terms of neighborliness of the symmetric convex hull of $X_1, \dots, X_N$, as was shown in [7].

Let T be an arbitrary $n \times N$ matrix. For $1 \leqslant m \leqslant N$, the isometry constant of T is defined as the smallest number $\delta_m = \delta_m(T)$ satisfying

$$(1 - \delta_m)|z|^2 \leqslant |Tz|^2 \leqslant (1 + \delta_m)|z|^2, \quad \text{for } z \in \mathbb{R}^N \text{ with } \big|\operatorname{supp}(z)\big| \leqslant m. \tag{2}$$

Let $\delta \in (0, 1)$. The matrix T is said to satisfy the RIP of order m with parameter δ if $0 \leqslant \delta_m(T) < \delta$.

Returning to the vectors $X_1, \dots, X_N$ (independent and satisfying (1)) the concentration of $|X_i|$'s around their average is controlled by the function

$$P(\theta) := \mathbb{P}\left(\max_{i \leqslant N}\left|\frac{|X_i|^2}{n} - 1\right| \geqslant \theta\right) \quad \text{for } \theta \in (0, 1). \tag{3}$$

Note that in order to have RIP, we need $P(\theta) < 1$, as the maximum under the probability equals to $\delta_1(A/\sqrt{n})$, which is less than or equal to $\delta_m(A/\sqrt{n})$.

Conditions saying that a random matrix satisfies RIP were investigated in many works. We refer to [8] and references therein. In [3] the authors studied the model when a random matrix consists of independent columns. It was proved that if X_i's are centered of variance 1 and satisfy assumption (1) with $\phi(t) = (1/2)\exp(t^\alpha)$, $\alpha \geqslant 1$, then A satisfies RIP with high probability. However, due to technical reasons, the case $\alpha < 1$ was left open. Moreover, it was not clear if RIP can hold under assumptions on moments of X_i's. In this note, we show that A satisfies RIP not only in the case $\alpha < 1$, but also when the marginals of X_i's satisfy moments condition only, that is condition (1) with $\phi(t) = t^p$, $p > 4$. Note that in view of a result by Bai, Silverstein and Yin [5], it seems that one cannot expect similar bounds when $p < 4$.

Theorem 1. Case 1. *Let $p > 4$ and $\phi(t) = t^p$. Let $0 < \varepsilon < \min\{1, (p-4)/4\}$ and $0 < \theta < 1$. Assume that $2^8/(\varepsilon\theta) \leqslant N \leqslant c\theta(c\varepsilon\theta)^{p/2}n^{p/4}$ and set*

$$m = \left[C(\varepsilon, p)\theta^{2p/(p-4-2\varepsilon)} n\left(\frac{N}{n}\right)^{-2(2+\varepsilon)/(p-4-2\varepsilon)}\right].$$

Case 2. *Let $\phi(t) = (1/2)\exp(t^\alpha)$. Assume that $8/\theta \leqslant N \leqslant c\theta\exp((1/2)(c\theta\sqrt{n})^\alpha)$ and set*

$$m = \left[C^{-2/\alpha}\theta^2 n\big(\ln\big(C^{2/\alpha}N/\big(\theta^2 n\big)\big)\big)^{-2/\alpha}\right].$$

In both cases, we have $\mathbb{P}(\delta_m(A/\sqrt{n}) \leqslant \theta) \geqslant 1 - 2^{-9}\theta - P(\theta/2)$.

3. Kannan–Lovász–Simonovits problem (KLS)

Let X_i's and A be as above and assume additionally that X_i's are identically distributed as a centered random vector X. KLS problem asks how fast the empirical covariance matrix $T := (1/N)AA^\top$ converges to the covariance matrix $\Sigma := (1/N)\mathbb{E}AA^\top$ (originally it was asked about so-called log-concave random vectors). In particular, is it true that with high probability, the operator norm $\|T - \Sigma\| \leqslant \varepsilon\|\Sigma\|$ for N being proportional to n? The corresponding important question in Random Matrix Theory is about the limit behavior of smallest and largest singular values. In the case of Wishart matrices, that is when the coordinates of X are i.i.d. random variables with finite fourth moment, the Bai–Yin theorem [4] states that the limits of minimal and maximal singular numbers of T are $(1 \pm \sqrt{\beta})^2$ as $n, N \to \infty$ and $n/N \to \beta \in (0, 1)$. Moreover, it is known [5] that boundedness of fourth moment is needed in order to have the convergence of the largest singular value. The asymptotic non-limit behavior (also called "non-asymptotic" in Statistics), i.e., sharp upper and lower bounds for singular