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Revisiting the Leinster groups

Quelques résultats sur les groupes de Leinster

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ABSTRACT

A finite group is said to be a Leinster group if the sum of the orders of its normal subgroups equals twice the order of the group itself. In this paper we give some new results concerning Leinster groups.

RÉSUMÉ

Un groupe de Leinster est un groupe fini tel que la somme des cardinaux de ses sous-groupes distingués soit égale au double du cardinal de G. Dans cette note, nous donnons quelques résultats nouveaux sur les groupes de Leinster.

1. Introduction

A number is perfect if the sum of its divisors equals twice the number itself. In 2001, T. Leinster [6], developed and studied a group-theoretic analogue of perfect numbers. A finite group is said to be a perfect group (not to be confused with the one which is equal to its commutator subgroup) or an immaculate group or a Leinster group if the sum of the orders of its normal subgroups equals twice the order of the group itself. Clearly, a finite cyclic group C_n is Leinster if and only if its order n is a perfect number. In fact, the abelian Leinster groups are precisely the finite cyclic groups whose orders are perfect numbers. It may be mentioned here that up to now, only one Leinster group of odd order is known, namely $(C_{127} \rtimes C_7) \times C_{3^4.11^2.19^2.113}$. It was discovered by F. Brunault [8]. More information on this and the related concepts can be found in the works of S.J. Baishya and A.K. Das [3], A.K. Das [4], M. Tărnăuceanu [10,12], T.D. Medts and A. Maróti [9], etc.

Given a finite group G, let $\tau(G)$ denote the number of normal subgroups of G and $\sigma(G)$ denote the sum of the orders of the normal subgroups of G. In this paper, among other results, we classify Leinster groups G with $\tau(G) \leqslant 7$.

2. Some basic results

The Leinster groups among the dihedral groups are in one-to-one correspondence with the odd perfect numbers [6, Example 2.4] and so it is an open question as to whether there are any. It would be interesting to find the Leinster groups among the well-known families of groups. In the following result, we classify the Leinster groups among the generalized quaternion group Q_{4m} of order $4m$, $m \geqslant 2$ given by $\langle a, b \mid a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1} \rangle$.

Proposition 2.1. *The generalized quaternion group Q_{4m}, $m \geqslant 2$ is Leinster if and only if $m = 3$.*

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Proof. It is well known that $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$, for any integer $m \geqslant 2$. By [9, Observation 3.1], we have $\frac{\sigma(Q_{4m})}{|Q_{4m}|} \geqslant \frac{\sigma(D_{2m})}{|D_{2m}|}$. Now, if m is even, then by [6, Example 2.4], we have $\frac{\sigma(D_{2m})}{|D_{2m}|} > 2$ and so Q_{4m} is not Leinster. Next, suppose m is odd. In this situation, it can be easily proved that the proper normal subgroups of Q_{4m} are precisely the subgroups of the cyclic group generated by a. Consequently, $\sigma(Q_{4m}) = 4m + \sigma(2m)$, where $\sigma(2m)$ is the sum of the positive divisors of $2m$. We know that 6 is the only perfect number of the form $2m$, where m is odd. Therefore Q_{4m}, $m \geqslant 2$ is Leinster if and only if $m = 3$. $\square$

Let $I(G)$ denote the set of all solutions of the equation $x^2 = 1$ in G. If n is an odd perfect number, then D_{2n} is a Leinster group [6, Example 2.4] with $|I(D_{2n})| > n$. The following theorem shows that if G is a Leinster group such that $|I(G)| > \frac{|G|}{2}$, then $G \cong D_{2n}$, where n is an odd perfect number. In the following theorem $|Cent(G)|$ denotes the number of distinct centralizers of G. Recall that a finite group G is said to be a CA-group if the centralizer $C(x)$ is abelian for every $x \in G \setminus Z(G)$ (see [5]).

Theorem 2.2. *If G is a Leinster group with $|I(G)| > \frac{|G|}{2}$, then $G \cong D_{2n}$, where n is an odd perfect number.*

Proof. Let G be a Leinster group with $|I(G)| > \frac{|G|}{2}$. Clearly, $|G|$ is even, otherwise $|I(G)| = 1$ and hence G is trivial, which is not possible. Now, suppose G is abelian. Then G is cyclic by [6, Corollary 4.2] and hence $|I(G)| = 2$. It follows that $|G| = 2$, which is not possible. Therefore, G is non-abelian and without any loss we can assume that $|G| = 2^m n$, where $m \geqslant 1$ is an integer and $n > 1$ is an odd integer, noting that a 2-group is not Leinster [6, Example 2.3]. Let G' be the commutator subgroup of G. By [14, p. 251], we have $\frac{G}{G'}$ is an elementary abelian 2-group and therefore by [6, Theorem 4.1], we have $|\frac{G}{G'}| = 2$. Again, by [14, Theorem 5], we have G' is abelian and hence by [2, Theorem 2.3], we get:

$$\left|Cent(G)\right| = \left|G'\right| + 2 = \frac{|G|}{2} + 2. \tag{1}$$

It is easy to see that $Z(G) \subsetneq G'$. Now, it follows from (1) that the elements of $G \setminus G'$ will produce exactly $\frac{|G|}{2}$ distinct centralizers, noting that elements of G' produce exactly two distinct centralizers, namely G and $G' = C(g)$, where $g \in G' \setminus Z(G)$. Let $a, b \in G \setminus G'$ such that $a \neq b$. Since $|G \setminus G'| = \frac{|G|}{2}$, it follows that:

$$C(a) \neq C(b). \tag{2}$$

Again, since G' is an abelian normal subgroup of G of index 2, therefore by [5, Theorem A], we have G is a CA-group and hence by [5, Proposition 3.2], we get:

$$ab \neq ba. \tag{3}$$

Now, suppose $|Z(G)| \neq 1$. Let $x \in G \setminus G'$. Since $Z(G) \subsetneq C(x)$, therefore by (3), there exists $y \in G' \setminus Z(G)$ such that $y \in C(x)$. It follows that $x \in C(y) = G'$, which is a contradiction. Hence $|Z(G)| = 1$. Again, by [14, Lemma 9], we have $I(G') \subseteq Z(G)$ and hence $|G'| = n$, noting that $|\frac{G}{G'}| = 2$.

Let $z \in G \setminus G'$. If $|C(z)| \neq 2$, then there exists $w \in C(z)$ such that $w \neq z$ and $w \neq 1$. It follows from (3) that $w \in G' \setminus Z(G)$. But then $z \in C(w) = G'$, which is a contradiction. Therefore $|C(z)| = 2$ and hence:

$$\left|Cl(z)\right| = \frac{|G|}{2}. \tag{4}$$

Now, suppose $N \trianglelefteq G$, $N \neq G$. If $|N|$ is even, then there exists $u \in N$ such that $o(u) = 2$. Clearly, $u \in G \setminus G'$ and hence by (4), we have $|Cl(u)| = \frac{|G|}{2}$. But $Cl(u) \subsetneq N$, which is not possible. Therefore $|N|$ is odd and hence $N \trianglelefteq G'$.

Again, note that $v \sim v^{-1}$ for any $v \in G'$. For if, $v \nsim v^{-1}$ for some $v \in G'$ then the map:

$$\phi_v : I(G) \longrightarrow G \setminus I(G) \quad \text{given by } h \longmapsto hv$$

is one–one, which is not possible since $|I(G)| > \frac{|G|}{2}$.

Now, suppose $N \trianglelefteq G'$. Since G' is abelian of index 2, therefore $Cl(r) = \{r, r^{-1}\}$ for any $r \in G' \setminus Z(G)$. Hence $N \trianglelefteq G$. Therefore, the proper normal subgroups of G are precisely the normal subgroups of G'. In other words, $\sigma(G) = |G| + \sigma(G')$. Now, since G is a Leinster group, it follows that $|G| = 2|G'| = \sigma(G')$. Therefore by [6, Corollary 4.2], G' is cyclic and $|G'| = n$ is an odd perfect number. Hence $G \cong D_{2n}$. $\square$

A group is said to be semi-simple if it is a direct product of non-abelian simple groups. In this connection, we have the following result.

Proposition 2.3. *No finite semi-simple group is Leinster.*