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Maximum principles and isoperimetric inequalities for some Monge–Ampère-type problems

Principes du maximum et inégalités isopérimétriques pour certains problèmes du type Monge–Ampère

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A R T I C L E   I N F O

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A B S T R A C T

In this note we derive a maximum principle for an appropriate functional combination of $u(\mathbf{x})$ and $|\nabla u|^2$, where $u(\mathbf{x})$ is a strictly convex classical solution to a general class of Monge–Ampère equations. This maximum principle is then employed to establish some isoperimetric inequalities of interest in the theory of surfaces of constant Gauss curvature in $\mathbb{R}^{N+1}$.

R É S U M É

Dans cette note, nous obtenons un principe du maximum pour une combinaison fonctionnelle appropriée de $u(\mathbf{x})$ et $|\nabla u|^2$, où $u(\mathbf{x})$ est une solution classique strictement convexe à une classe générale d'équations du type Monge–Ampère. Ce principe du maximum est ensuite utilisé pour établir certaines inégalités isopérimétriques d'intérêt dans la théorie de surfaces de courbure de Gauss constante dans $\mathbb{R}^{N+1}$.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded strictly convex C^2 domain. This note deals with the following general class of Monge–Ampère equations:

$$\det(D^2u) = f(u)g(|\nabla u|^2) \quad \text{in } \Omega, \tag{1.1}$$

where f and g are some real positive C^1 functions, with $f' \geqslant 0$. We also assume that Eq. (1.1) is uniformly elliptic, i.e. we impose throughout the strong ellipticity condition $A^* := (S_N^{ij}) > 0$ in Ω, where $S_N^{ij} := \partial(\det(D^2u))/\partial u_{ij}$, which means that a solution to Monge–Ampère equation (1.1) is assumed throughout to be strictly convex in Ω. Under these assumptions, Hopf's first maximum principle [5] implies that a classical solution $u(\mathbf{x})$ to Eq. (1.1) assumes its minimum value on $\partial\Omega$.

Let us now consider the following auxiliary function, which is a kind *P-function* in the sense of L.E. Payne (see the book of R. Sperb [15] or the paper of G.A. Philippin and S. Vernier-Piro [12]):

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$$P(\mathbf{x},\alpha) := \int_0^{|\nabla u|^2} \rho(y)\,dy - 2\alpha \int_0^u f(s)^{\frac{1}{N}}\,ds, \quad \text{with } \alpha \in [0,1], \tag{1.2}$$

where $u(\mathbf{x})$ is a classical solution of Eq. (1.1) and:

$$\rho(y) := \frac{1}{g(y)}\left(\frac{N}{2}y^{-\frac{N}{2}} \int_0^y \frac{1}{g(s)} s^{\frac{N}{2}-1}\,ds\right)^{\frac{1}{N}-1}. \tag{1.3}$$

The main result of this note states:

Theorem 1.1. *Assume that $u(\mathbf{x}) \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is a strictly convex solution of Eq. (1.1) and $P(\mathbf{x},\alpha)$ is the auxiliary function defined in (1.2). Then:*

i) *$P(\mathbf{x},0)$ takes its maximum value on $\partial\Omega$;*
ii) *if $g' \geqslant 0$ and $\alpha \in (0,1]$, $P(\mathbf{x},a)$ takes its maximum value on $\partial\Omega$.*

We note that particular cases of Theorem 1.1 have been considered and investigated in some previous works, namely in X.-N. Ma [8] (the case $N = 2$, $f \equiv g \equiv 1$; see, also, C. Enache [3] for a complementary result), G.A. Philippin and A. Safoui [11] (the case $f \equiv g \equiv 1$), C. Enache [4] (the case $N = 2$), respectively. Furthermore, G.A. Philippin and A. Safoui [10] (see, also, C. Enache [2] or L. Barbu and C. Enache [1]) have also investigated the general class of Monge–Ampère equations (1.1) and derived a similar maximum principle for a different auxiliary function, namely for:

$$\Phi(\mathbf{x}) := \int_0^{|\nabla u|^2} g^{-\frac{1}{N}}(s)\,ds - 2\int_0^u f^{\frac{1}{N}}(s)\,ds, \tag{1.4}$$

under the following additional assumption on the data f and g:

$$g^{-\frac{1}{N}}\frac{f'}{f} + 2f^{\frac{1}{N}}\frac{g'}{g} \geqslant 0. \tag{1.5}$$

More precisely, they proved that $\Phi(\mathbf{x})$ *takes its maximum value on* $\partial\Omega$. However, as we will notice later (see Remark 1 in Section 2), the maximum principle stated in Theorem 1.1 for $P(\mathbf{x},1)$ is the best possible when $f \equiv$ *const*. This means that $P(\mathbf{x},1)$ satisfies a maximum principle and that there exists a domain of optimality on which P is identically constant. Thus, when Eq. (1.1) is subject to a Dirichlet boundary condition and $f \equiv$ *const*., our maximum principle may be employed to derive isoperimetric inequalities. Conversely, $\Phi(\mathbf{x})$ cannot be identically constant for $f \not\equiv$ *const*. or $g \not\equiv$ *const*., so that the maximum principle derived by G.A. Philippin and A. Safoui [10] is not the best possible in such a case.

As applications of Theorem 1.1, we are going to establish some isoperimetric inequalities of interest in the theory of surfaces of constant Gauss curvature. To this end, we will investigate the particular case $f \equiv k_0 = const. > 0$, $g(s) = (1+s)^{(N+2)/2}$ in (1.1), when k_0 represents the Gauss curvature of the hypersurface $x_{N+1} = u(x_1,\dots,x_N)$ in the Euclidean space $\mathbb{R}^{N+1}$. More precisely, let us consider the following problem:

$$\begin{cases} \det(D^2u) = k_0\left(1+|\nabla u|^2\right)^{\frac{N+2}{2}} > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Moreover, assume that the (Gauss) curvature $G(s)$ of $\partial\Omega$ satisfies the following existence criterion (see N.M. Ivochkina [7]):

$$G \geqslant k_0^{\frac{N-1}{N}} \quad \text{on } \partial\Omega. \tag{1.7}$$

Our second result states:

Theorem 1.2. *Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be the strictly convex solution of problem (1.6) and denote*

$$u_{\min} := \min_{\Omega} u(\mathbf{x}), \qquad S := \left\{\left(\mathbf{x}, u(\mathbf{x})\right) : \mathbf{x} = (x_1,\dots,x_N) \in \Omega\right\},$$

$$A := \int_\Omega \sqrt{1+|\nabla u|^2}\,dx \quad (\textit{the area of the surface } S),$$

$$V := -\int_\Omega u\,dx \quad (\textit{the volume between } \Omega \textit{ and the surface } S). \tag{1.8}$$