C. R. Acad. Sci. Paris, Ser. I 353 (2015) 11–15

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex analysis

A continuous link between the disk and half-plane cases of Grace's theorem

Un lien continu entre les cas du disque et du demi-plan dans le théorème de Grace

Martin Lamprecht

Department of Computer Science and Engineering, European University of Cyprus, Diogenous Str. 6, Engomi, P.O. Box 22006, 1516 Nicosia, Cyprus

ARTICLE INFO

Article history:
Received 29 September 2014
Accepted after revision 23 October 2014
Available online 4 November 2014

Presented by Jean-Pierre Kahane

ABSTRACT

We obtain a continuous link between the disk and half-plane cases of Grace's theorem and new, non-circular zero domains that stay invariant under the Schur–Szegő convolution.

RÉSUMÉ

On obtient un lien continu entre les cas du disque et du demi-plan dans le théorème de Grace, ainsi que de nouveaux domaines de zéros non cerclés, qui sont invariants par la convolution de Schur–Szegő.

1. Introduction

1.1. Main results

Let Ω be a connected set in $\mathbb{C}$. Depending on whether Ω is bounded or unbounded, we denote by $\pi_n(\Omega)$ the set of all polynomials of degree n or $\leq n$ with zeros only in Ω. A polynomial $g(z) = \sum_{k=0}^{n} b_k z^k$ of degree n is called a *multiplier* of $\pi_n(\Omega)$ if the *convolution*

$$(f * g)(z) := \sum_{k=0}^{n} a_k b_k z^k$$

of g with every $f(z) = \sum_{k=0}^{n} a_k z^k$ in $\pi_n(\Omega)$ again belongs to $\pi_n(\Omega)$. We denote the set of multipliers of $\pi_n(\Omega)$ by $\mathcal{M}_n(\Omega)$. The *pre-coefficient class* $\pi_n^*(\Omega)$ of a connected set $\Omega \subset \mathbb{C}$ is the set of all polynomials $f(z) = \sum_{k=0}^{n} b_k z^k$ for which

E-mail address: m.lamprecht@euc.ac.cy.

http://dx.doi.org/10.1016/j.crma.2014.10.017
1631-073X/

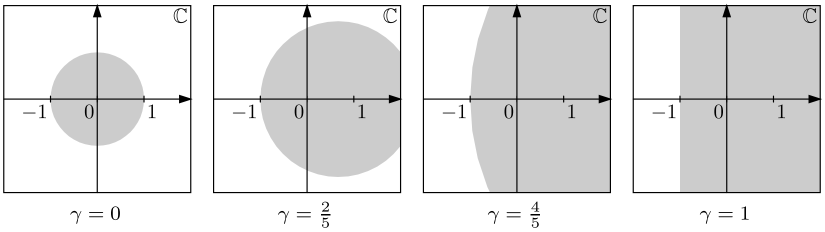

Fig. 1. The sets $\Omega_{-(1+\gamma),\gamma}$ (grey area) for certain values of γ.

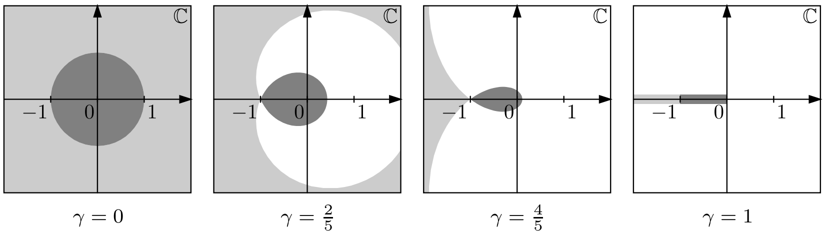

Fig. 2. The sets $\bar{I}_\gamma$ (dark grey) and $\overline{O}_\gamma$ (light grey) for certain values of γ.

$$f(z) * (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} b_k z^k \in \pi_n(\Omega).$$

In this note we show that for every open or closed disk $\Omega \subset \mathbb{C}$ that contains the origin in its interior there is an associated set $\Omega^* \subset \mathbb{C}$ such that $\mathcal{M}_n(\Omega) = \pi_n^*(\Omega^*)$.

In order to give an explicit description of the sets Ω^*, note that, as explained in [5], for every open disk or half-plane Ω that contains the origin, there are two unique parameters $\tau \in \mathbb{C} \setminus \{0\}$ and $\gamma \in [0,1]$ such that Ω is the image of the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ under a Möbius transformation of the form

$$w_{\tau,\gamma}(z) := \frac{\tau z}{1+\gamma z}.$$

We write $\Omega_{\tau,\gamma}$ for such a domain and note that, for all $\tau \in \mathbb{C} \setminus \{0\}$ (cf. also Fig. 1),

$$\Omega_{\tau,0} = \left\{z \in \mathbb{C} : |z| < |\tau|\right\} \quad \text{and} \quad \Omega_{\tau,1} = \left\{z \in \mathbb{C} : \Re\left(\tau^{-1} z\right) < \frac{1}{2}\right\}. \tag{1}$$

For $\gamma \in [0,1)$ we also define

$$I_\gamma := \left\{z \in \mathbb{C} : |z| + \gamma|1+z| < 1\right\} \quad \text{and} \quad O_\gamma := \left\{z \in \mathbb{C} : |z| - \gamma|1+z| > 1\right\}.$$

$\bar{I}_\gamma$ and $\overline{O}_\gamma$ are families of sets that, when γ increases from 0 to 1, decrease from $\bar{I}_0 = \overline{\mathbb{D}}$ and $\overline{O}_0 = \mathbb{C} \setminus \mathbb{D}$ to

$$\bar{I}_1 := \bigcap_{\gamma \in [0,1)} \bar{I}_\gamma = [-1, 0] \quad \text{and} \quad \overline{O}_1 := \bigcap_{\gamma \in [0,1)} \overline{O}_\gamma = (-\infty, -1], \tag{2}$$

respectively. For $\gamma \in (0,1)$, I_γ is the interior of the inner loop of the limaçon of Pascal, and O_γ is the open exterior of the limaçon of Pascal (cf. Fig. 2).

Our main result can now be stated as follows.

Theorem 1.1. *Let* $\tau \in \mathbb{C} \setminus \{0\}$ *and* $\gamma \in [0,1]$. *Then*

(i) $\mathcal{M}_n(\overline{\Omega}_{\tau,\gamma}) = \mathcal{M}_n(\Omega_{\tau,\gamma}) = \pi_n^*(\overline{I_\gamma})$, *and*
(ii) $\mathcal{M}_n(\mathbb{C} \setminus \Omega_{\tau,\gamma}) = \mathcal{M}_n(\mathbb{C} \setminus \overline{\Omega}_{\tau,\gamma}) = \pi_n^*(\overline{O}_\gamma)$.

By the definition of multiplier classes, it is clear that $f, g \in \mathcal{M}_n(\Omega)$ implies $f * g \in \mathcal{M}_n(\Omega)$. Theorem 1.1 thus leads to the following description of $\mathcal{M}_n(\Omega)$ for the domains $\Omega = I_\gamma$ and $\Omega = O_\gamma$.

Corollary 1.2. *Let* $\gamma \in [0,1)$. *Then*

$$\mathcal{M}_n(I_\gamma) = \mathcal{M}_n(\bar{I}_\gamma) = \pi_n^*(\bar{I}_\gamma) \quad \textit{and} \quad \mathcal{M}_n(O_\gamma) = \mathcal{M}_n(\overline{O}_\gamma) = \pi_n^*(\overline{O}_\gamma).$$