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Harmonic analysis

On irregular sampling in Bernstein spaces

Sur l'échantillonnage irrégulier dans les espaces de Bernstein

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ABSTRACT

We obtain sharp estimates for the sampling constants in Bernstein spaces when the density of the sampling set is near the critical value.

RÉSUMÉ

Nous obtenons des estimations finales pour les constantes de l'échantillonnage dans les espaces de Bernstein lorsque la densité des échantillons est proche de la valeur critique.

1. Introduction

Given a number $\sigma > 0$, the Bernstein space B_σ is defined to be the set of all entire functions f satisfying for all real x and y the inequality $|f(x+iy)| \le C \exp(\sigma|y|)$ with some $C = C(f)$.

A set $\Lambda \subset \mathbb{R}$ is called uniformly discrete (u.d.) if

$$\inf_{\lambda,\lambda'\in\Lambda,\lambda\neq\lambda'} \left|\lambda-\lambda'\right| > 0.$$

One says that Λ is a (stable) sampling set for B_σ if there exists K such that

$$\|f\| := \sup_{t\in\mathbb{R}} \left|f(t)\right| \le K \sup_{\lambda\in\Lambda} \left|f(\lambda)\right| \quad (f \in B_\sigma).$$

The minimal constant K for which this holds is called the sampling constant $K(\Lambda, B_\sigma)$.

The classical Beurling theorem [2] characterizes sampling sets for B_σ in terms of the lower uniform density

$$D^-(\Lambda) := \lim_{l\to\infty} \min_{a\in\mathbb{R}} \frac{\#\Lambda \cap (a, a+l)}{l}.$$

Without loss of generality, one may consider the case $\sigma = \pi$. Then Beurling's theorem states:

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Λ is a sampling set for B_π if and only if $D^-(\Lambda) > 1$.

The most delicate point in Beurling's proof (see [2]) is to show that no sampling set Λ may have the critical density $D^-(\Lambda) = 1$.

If $D^-(\Lambda) = 1$, one can show that constant $K(\Lambda, B_\sigma)$ grows to infinity when σ approaches 1 from below. When $\Lambda = \mathbb{Z}$, S.N. Bernstein [1] proved that *the growth is precisely logarithmic:*

$$K(\mathbb{Z}, B_\sigma) = \frac{2}{\pi} \log \frac{\pi}{\pi - \sigma}\big(1 + o(1)\big) \quad (\sigma \uparrow \pi).$$

A slightly weaker result was proved in [3]. See also [6] where some estimates for $K(\Lambda, B_\sigma)$ are obtained. We mention also [4], where the Gabor frame considered for the Gaussian window, which corresponds to the lattice $a\mathbb{Z} \times a\mathbb{Z}$, and the asymptotics of the frames constants are obtained near the critical value $a = 1$.

2. Results

2.1. Sampling in Bernstein spaces

We are interested in the asymptotic behavior of the sampling constant $K(\Lambda, B_\sigma)$ for irregular sampling Λ near the critical value of density. Our main result shows that $K(\Lambda, B_\sigma)$ must have at least logarithmic growth.

We will denote by C different absolute positive constants.

Theorem 1. *Let Λ be a u.d. set with $D^-(\Lambda) = 1$. Then*

$$K(\Lambda, B_\sigma) \ge C \log \frac{\pi}{\pi - \sigma} \quad (0 < \sigma < \pi). \tag{1}$$

The proof is based on a reduction of the sampling problem to a similar one for the algebraic polynomials. This approach provides a new proof for the critical case in Beurling's theorem above.

It should be mentioned that removing even a single point from Λ may result in a much faster growth of the sampling constants. For example, it is straightforward to check that

$$K\big(\mathbb{Z} \setminus \{0\}, B_\sigma\big) \ge \frac{\sigma}{\pi - \sigma} \quad (0 < \sigma < \pi).$$

In fact, the constant $K(\Lambda, B_\sigma)$ may have arbitrarily fast growth:

Theorem 2. *For every function $\omega(\sigma) \uparrow \infty$ ($\sigma \uparrow \pi$) there exists a u.d. set Λ, $D^-(\Lambda) = 1$, such that*

$$K(\Lambda, B_\sigma) \ge \omega(\sigma) \quad (\sigma < \pi).$$

2.2. Sampling in P_n

Denote by P_n the space of all algebraic polynomials of degree $\le n$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Given a finite set $\Lambda \subset \mathbb{T}$, $\#\Lambda > n$, one may introduce the corresponding sampling constant

$$K(\Lambda, P_n) := \sup_{P \in P_n, P \ne 0} \frac{\max_{z \in \mathbb{T}} |P(z)|}{\max_{\lambda \in \Lambda} |P(\lambda)|}.$$

Theorem 3. *For every $\Lambda \subset \mathbb{T}$, $\#\Lambda > n$, the estimate holds:*

$$K(\Lambda, P_n) \ge C \log \frac{n}{\#\Lambda - n}. \tag{2}$$

3. Sampling in spaces of polynomials

The following result essentially goes back to Faber:

Let U be a projector from the space $C(\mathbb{T})$ onto the subspace P_n. Then $\|U\| > C \log n$,

see [5], ch. 7.

Faber's approach is based on averaging over translations. Different versions of the result have been obtained by this approach. We will use the following one due to Al.A. Privalov [8] (see also [7]):

For every projector U above and every family of linear functionals ψ_j $(1 \le j \le m)$ in $C(\mathbb{T})$, there is a unit vector f in $C(\mathbb{T})$ such that $\|Uf\| > C \log n/m$, and the functionals vanish on f.