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Mathematical Problems in Mechanics

Expression of Dirichlet boundary conditions in terms of the strain tensor in linearized elasticity

Expression de conditions aux limites de Dirichlet en fonction du tenseur linéarisé des déformations

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A R T I C L E I N F O

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A B S T R A C T

In a previous work, it was shown how the linearized strain tensor field $\mathbf{e} := \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}) \in \mathbb{L}^2(\Omega)$ can be considered as the sole unknown in the Neumann problem of linearized elasticity posed over a domain $\Omega \subset \mathbb{R}^3$, instead of the displacement vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ in the usual approach. The purpose of this Note is to show that the same approach applies as well to the Dirichlet–Neumann problem. To this end, we show how the boundary condition $\boldsymbol{u} = \mathbf{0}$ on a portion Γ_0 of the boundary of Ω can be recast, again as boundary conditions on Γ_0, but this time expressed only in terms of the new unknown $\mathbf{e} \in \mathbb{L}^2(\Omega)$.

R É S U M É

Dans un travail antérieur, on a montré comment le champ $\mathbf{e} := \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}) \in \mathbb{L}^2(\Omega)$ des tenseurs linéarisés des déformations peut être considéré comme la seule inconnue dans le problème de Neumann pour l'élasticité linéarisée posé sur un domaine $\Omega \subset \mathbb{R}^3$, au lieu du champ $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ des déplacements dans l'approche habituelle. L'objet de cette Note est de montrer que la même approche s'applique aussi bien au problème de Dirichlet–Neumann. À cette fin, nous montrons comment la condition aux limites $\boldsymbol{u} = \mathbf{0}$ sur une portion Γ_0 de la frontière de Ω peut être ré-écrite, à nouveau sous forme de conditions aux limites sur Γ_0, mais exprimées cette fois uniquement en fonction de la nouvelle inconnue $\mathbf{e} \in \mathbb{L}^2(\Omega)$.

1. Preliminaries

Greek indices, resp. Latin indices, range over the set $\{1, 2\}$, resp. $\{1, 2, 3\}$. The summation convention with respect to repeated indices is used in conjunction with these rules. The notations $|\mathbf{a}|$, $\mathbf{a} \wedge \boldsymbol{b}$, $\mathbf{a} \otimes \boldsymbol{b}$, and $\mathbf{a} \cdot \boldsymbol{b}$ respectively denote the Euclidean norm, the exterior product, the dyadic product, and the inner product of vectors $\mathbf{a}, \boldsymbol{b} \in \mathbb{R}^3$.

The notation $\mathbb{S}^m$, resp. $\mathbb{A}^m$, designates the space of all symmetric, resp. antisymmetric, tensors of order m. The inner product of two $m \times m$ tensors $\mathbf{e}$ and $\boldsymbol{\tau}$ is denoted and defined by $\mathbf{e} : \boldsymbol{\tau} = \operatorname{tr}(\mathbf{e}^T \boldsymbol{\tau})$. Given a normed vector space X, the notation $\mathcal{L}^2_{\text{sym}}(X \times X)$ designates the space of all continuous symmetric bilinear forms defined on the product $X \times X$.

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Let $\Omega \subset \mathbb{R}^3$ be a connected, bounded, open set whose boundary $\partial\Omega$ is of class $\mathcal{C}^4$. This means that there exist a finite number N of open sets $\omega^k \subset \mathbb{R}^2$ and of mappings $\boldsymbol{\theta}^k \in \mathcal{C}^4(\omega^k; \mathbb{R}^3)$, $k = 1, 2, \dots, N$, such that $\partial\Omega = \bigcup_{k=1}^N \boldsymbol{\theta}^k(\omega^k)$. It also implies that there exists $\varepsilon > 0$ such that the mappings $\boldsymbol{\Theta}^k \in \mathcal{C}^3(U^k; \mathbb{R}^3)$, defined by:

$$\boldsymbol{\Theta}^k(y, y_3) := \boldsymbol{\theta}^k(y) + y_3 \mathbf{a}_3^k(y) \quad \text{for all } (y, y_3) \in U^k := \omega^k \times (-\varepsilon, \varepsilon),$$

where $\mathbf{a}_3^k$ denotes the unit inner normal vector field along the portion $\boldsymbol{\theta}^k(\omega^k)$ of the boundary of Ω, are $\mathcal{C}^3$-diffeomorphisms onto their image (cf. [2, Theorem 4.1-1]). Thus the mappings $\{\boldsymbol{\Theta}^k;\ 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the open set $\Omega_\varepsilon := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < \varepsilon\} \subset \mathbb{R}^3$, while the mappings $\{\boldsymbol{\theta}^k; 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the surface $\Gamma = \partial\Omega \subset \mathbb{R}^3$. When no confusion should arise, we will drop the explicit dependence on k for notational brevity.

A generic point in ω is denoted $y = (y_\alpha)$ and a generic point in $U = \omega \times (-\varepsilon, \varepsilon)$ is denoted (y, y_3). Partial derivatives with respect to y_i are denoted ∂_i. The vectors $\mathbf{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ form a basis in the tangent space at $\boldsymbol{\theta}(y)$ to the surface $\Gamma := \partial\Omega \subset \mathbb{R}^3$ and the vectors $\mathbf{g}_i(y, y_3) := \partial_i \boldsymbol{\Theta}(y, y_3)$ form a basis in the tangent space at $\boldsymbol{\Theta}(y, y_3)$ to the open set $\boldsymbol{\Theta}(U) \subset \Omega_\varepsilon \subset \mathbb{R}^3$. Note that:

$$\mathbf{g}_\alpha(y, y_3) = \mathbf{a}_\alpha(y) + y_3 \partial_\alpha \mathbf{a}_3(y) \quad \text{and} \quad \mathbf{g}_3(y, y_3) = \mathbf{a}_3(y).$$

By exchanging if necessary the coordinates y_1 and y_2, we may always assume that:

$$\mathbf{a}_3(y) = \frac{\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)|}.$$

The vectors $\mathbf{a}^\alpha(y)$ in the tangent space at $\boldsymbol{\theta}(y)$ to Γ and $\mathbf{g}^i(y, y_3)$ in the tangent space at $\boldsymbol{\Theta}(y, y_3)$ are defined by:

$$\mathbf{a}^\alpha(y) \cdot \mathbf{a}_\beta(y) = \delta^\alpha_\beta \quad \text{and} \quad \mathbf{g}^i(y, y_3) \cdot \mathbf{g}_j(y, y_3) = \delta^i_j,$$

the area element on Γ is $\mathrm{d}\Gamma := \sqrt{a}\mathrm{d}y$, where $a := |\mathbf{a}_1 \wedge \mathbf{a}_2|$, and the Christoffel symbols $C^\sigma_{\alpha\beta}$ and Γ^k_{ij}, respectively induced by the immersions $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$, are defined by:

$$C^\sigma_{\alpha\beta} := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \mathbf{a}^\sigma \quad \text{and} \quad \Gamma^k_{ij} := \partial_{ij} \boldsymbol{\Theta} \cdot \mathbf{g}^k.$$

A point in Ω will be specified either by its Cartesian coordinates $x = (x_i)$ with respect to a given orthonormal basis $\hat{\mathbf{e}}^i$ in $\mathbb{R}^3$, or, when $x \in \Omega_\varepsilon \subset \Omega$, by its curvilinear coordinates (y, y_3) corresponding to a local chart $\boldsymbol{\Theta}$; thus $x = \boldsymbol{\Theta}(y, y_3)$ in such a local chart.

Vector fields, resp. tensor fields, on Ω will be expanded at each $x = \boldsymbol{\Theta}(y, y_3) \in \Omega_\varepsilon$ over the contravariant bases $\mathbf{g}^i(y, y_3)$, resp. $(\mathbf{g}^i \otimes \mathbf{g}^j)(y, y_3)$. Covariant derivatives with respect to the local chart $\boldsymbol{\Theta}$ are defined as usual, and denoted $u_{i\|j}$, $u_{i\|jk}$, $e_{ij\|k}$, etc.

Let Γ_0 be a connected and relatively open subset of the boundary Γ of Ω. Since Γ is a manifold of class $\mathcal{C}^4$, so is Γ_0. It follows that functions, vector fields, and tensor fields, of class $\mathcal{C}^m$, $m = 0, 1, 2$, can be defined on Γ_0. The Lebesgue and Sobolev spaces on Γ_0 and their norms are then defined as in, e.g., Aubin [1].

We also let $\mathcal{C}^m_c(\Gamma_0)$ denote the space of all functions $f : \Gamma_0 \to \mathbb{R}$ of class $\mathcal{C}^m$ with compact support contained in Γ_0. Then the Sobolev space $H^m_0(\Gamma_0)$ is defined as the completion of the space $\mathcal{C}^m_c(\Gamma_0)$ with respect to the norm $\|\cdot\|_{H^m(\Gamma_0)}$. Its dual space is denoted $H^{-m}(\Gamma_0)$.

Spaces of vector fields, resp. symmetric tensor fields, with values in $\mathbb{R}^3$, resp. in $\mathbb{S}^3$, are defined by using a given Cartesian basis $\{\hat{\mathbf{e}}_i, 1 \leqslant i \leqslant 3\}$ in $\mathbb{R}^3$, resp. the basis $\{\frac{1}{2}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j + \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i), 1 \leqslant i, j \leqslant 3\}$ in $\mathbb{S}^3$. They will be denoted by bold letters and by capital Roman letters, respectively.

Complete proofs and complements will be found in [5].

2. Linearized change of metric and curvature tensors on $\partial\Omega$ associated with a linearized strain tensor in $\mathbb{C}^1(\overline{\Omega})$

Given any displacement field $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$, the restriction $\boldsymbol{\zeta} := \boldsymbol{u}|_{\overline{\Gamma}_0} \in \boldsymbol{\mathcal{C}}^2(\overline{\Gamma}_0)$ is a displacement field of the surface $\overline{\Gamma}_0 \subset \mathbb{R}^3$. The linearized change of metric and change of curvature tensor fields induced by $\boldsymbol{\zeta}$ are then respectively defined in each local chart by:

$$\begin{aligned}
\boldsymbol{\gamma}(\boldsymbol{\zeta}) &= \gamma_{\alpha\beta}(\boldsymbol{\zeta}) \mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \gamma_{\alpha\beta}(\boldsymbol{\zeta}) := \frac{1}{2}(\partial_\alpha \boldsymbol{\zeta} \cdot \mathbf{a}_\beta + \partial_\beta \boldsymbol{\zeta} \cdot \mathbf{a}_\alpha), \\
\boldsymbol{\rho}(\boldsymbol{\zeta}) &= \rho_{\alpha\beta}(\boldsymbol{\zeta}) \mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \rho_{\alpha\beta}(\boldsymbol{\zeta}) := \big(\partial_{\alpha\beta} \boldsymbol{\zeta} - C^\sigma_{\alpha\beta} \partial_\sigma \boldsymbol{\zeta}\big) \cdot \mathbf{a}_3,
\end{aligned} \tag{1}$$

where for convenience the *same* notation $\boldsymbol{\zeta}$ denotes *either* the vector field $\boldsymbol{\zeta} : \overline{\Gamma}_0 \to \mathbb{R}^3$ *or* the vector field $\boldsymbol{\zeta} := \boldsymbol{\zeta} \circ \boldsymbol{\theta} : \omega \to \mathbb{R}^3$ in a local chart $\boldsymbol{\theta} : \omega \to \overline{\Gamma}_0$ of $\overline{\Gamma}_0$.

Let $T_x\Gamma_0 \subset \mathbb{R}^3$ denote the tangent space at each point x of the surface Γ_0. Given any matrix field $\mathbf{e} \in \mathbb{C}^1(\overline{\Omega})$, let the tensor fields: