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Group theory

Finite metacyclic groups as active sums of cyclic subgroups

Les groupes finis métacycliques comme sommes actives de sous-groupes cycliques

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ABSTRACT

The notion of active sum provides an analogue for groups of what the direct sum is for abelian groups. One natural question then is which groups are the active sum of a family of cyclic subgroups. Many groups have been found to give a positive answer to this question, while the case of finite metacyclic groups remained unknown. In this note we show that every finite metacyclic group can be recovered as the active sum of a discrete family of cyclic subgroups.

RÉSUMÉ

La notion de somme active fournit un analogue pour les groupes de ce qu'est la somme directe pour les groupes abéliens. Une question naturelle est alors de déterminer quels groupes sont la somme active d'une famille de sous-groupes cycliques. De nombreux groupes possèdent cette propriété, mais la question demeurait ouverte pour les groupes finis métacycliques. Dans cette note, nous montrons que tout groupe fini métacyclique s'obtient comme la somme active d'une famille discrète de sous-groupes cycliques.

1. Introduction

The active sum of groups has its origin in a paper of Tomás [10], where one of the main motivations was to find an analogue of the direct sum of groups, but this time taking into account the mutual actions of the groups in question. For an arbitrary group G, the active sum of a generating family of subgroups, closed under conjugation and with partial order compatible with inclusion, is a group S which has G as an homomorphic image and that coincides with the direct sum of the family in case of G being abelian. Since every finite abelian group is the direct sum of cyclic subgroups, it is natural to ask whether every finite group is the active sum of cyclic subgroups. In [2] it is shown that free groups, semidirect

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products of two finite cyclic groups, most of the groups of the form $SL_n(q)$, and Coxeter groups are all active sums of discrete families of cyclic subgroups. On the other hand, in [3] we gave examples of finite groups that are not active sums of cyclic subgroups: the alternating groups A_n for $n \geq 4$, many of the groups of the form $PSL_n(q)$, and others. However, at the time we were unable to determine whether every finite metacyclic group is the active sum of a discrete family of cyclic subgroups, a question we settle in this paper.

Dealing with the definition of active sum in concrete examples is not an easy task. Fortunately, one of its main properties is that for any group G and any family $\mathcal{F}$ satisfying the conditions mentioned before, the active sum S of $\mathcal{F}$ satisfies $S/Z(S) \cong G/Z(G)$ ($Z(S)$ and $Z(G)$ denote the center of S and G, respectively). This allows us to obtain conditions of homological nature, namely the surjectivity of Ganea's homomorphism, that helps us decide whether S is isomorphic to G. As we will see in Section 3, in case of G being finite metacyclic, this reduces the problem to verify that the family proposed is *regular* and *independent*. The notions of regularity and independence will be recalled in the next section, while Ganea's homomorphism will be described in Section 3.

A group G is *metacyclic* if it has a normal cyclic subgroup K such that G/K is also cyclic. For a finite metacyclic group G, Hyo-Seob Sim [9] introduces the *Standard Hall decomposition* of a given metacyclic factorization, splitting G as a semidirect product of two subgroups N and H of relatively prime order. The subgroup N turns out to be a semidirect product of two cyclic groups and H is nilpotent. By doing this, all the difficulties that arise when G is not a semidirect product of two cyclic groups are gathered in the nilpotent group H. This is precisely what we will use to show that every finite metacyclic group is an active sum of cyclic subgroups. As we mentioned above, we know this to be true for the semidirect product of two cyclic groups; we will prove it for finite metacyclic p-groups, and we will finally use the Standard Hall decomposition to prove it for any finite metacyclic group.

The active sum shares some nice properties with other similar constructions, such as the cellular cover of a group (see for example, [1] or [4]). In fact, it seems quite plausible that these two constructions are closely related. For example, if the family contains only cyclic groups of order n, one can show that the active sum is $\mathbb{Z}/n\mathbb{Z}$-cellular, in the sense of Definition 2.2 of [1]. As a consequence, we have that Coxeter groups are $\mathbb{Z}/2\mathbb{Z}$-cellular, since any Coxeter group is the active sum of a family of groups of order 2. For more information about the active sum, and its relation to other areas, see the introduction to [2].

2. Preliminaries

2.1. Active sum

We take as our definition of active sum the one given in [2]. Since in this paper we will only consider discrete families of distinct subgroups, for the convenience of the reader we briefly describe the results we need in this particular setting. Thus we take a (finite) group G, and a family $\mathcal{F}$ of distinct subgroups of G that is generating ($\langle \bigcup_{F\in\mathcal{F}} F\rangle = G$) and closed under conjugation ($\forall F \in \mathcal{F}$, $g \in G$, $F^g = g^{-1}Fg \in \mathcal{F}$). The *active sum* S of $\mathcal{F}$ is the free product of the elements of $\mathcal{F}$ divided by the normal subgroup generated by the elements of the form $h^{-1} \cdot g \cdot h \cdot (g^h)^{-1}$, with $h \in F_1$, $g \in F_2$, $F_1, F_2 \in \mathcal{F}$ (and thus, $g^h \in F_2^h = h^{-1}F_2h \in \mathcal{F}$). We obtain a canonical homomorphism $\varphi : S \to G$, surjective since $\mathcal{F}$ is generating. By Lemma 1.5 of [2], $\varphi^{-1}(Z(G)) = Z(S)$, so we obtain a central extension:

$$H_2(S) \xrightarrow{\varphi_*} H_2(G) \longrightarrow \ker\varphi \longrightarrow H_1(S) \xrightarrow{\varphi_*} H_1(G) \longrightarrow 1\,.$$

From Theorem 1.12 in [2] we have that $\varphi_* : H_1(S) \to H_1(G)$ is injective (and thus an iso) iff $\mathcal{F}$ is regular and independent, notions we will recall below. On the other hand, we will see in Section 3 that when G is a metacyclic group, the arrow $\varphi_* : H_2(S) \to H_2(G)$ is always surjective. As a consequence, in the particular case of G being metacyclic, $\varphi : S \to G$ is an isomorphism iff $\mathcal{F}$ is regular and independent.

The notions of regularity and independence are explained in more detail in Section 1.2 of [2]. For regularity we recall Lemma 1.9 of this reference:

Lemma 2.1. *Assume that the family $\mathcal{F}$ is discrete. The family $\mathcal{F}$ is regular iff $[F, N_G(F)] = F \cap G'$ for every $F \in \mathcal{F}$, where $N_G(F)$ is the normalizer of F in G.*

As for independence, we consider first a complete set of representatives $\mathcal{T}$ of conjugacy classes of elements of $\mathcal{F}$. We call $\mathcal{T}$ a transversal of $\mathcal{F}$. According to Definition 1.11 of [2], $\mathcal{F}$ is independent iff the canonical homomorphism

$$\bigoplus_{F\in\mathcal{T}} F/(F \cap G') \to G/G'$$

is an isomorphism.