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Transmutation operators associated with an integro-differential operator on the real line and certain of their applications

Opérateurs de transmutation associés à un opérateur intégro-différentiel sur la droite réelle et certaines de leurs applications

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A B S T R A C T

We consider a singular integro-differential operator Λ on the real line. We build transmutation operators of Λ and its dual $\widetilde{\Lambda}$ into the first derivative operator d/dx. Using these transmutation operators, we develop a new commutative harmonic analysis on the real line corresponding to the operator Λ.

R É S U M É

Nous considérons un opérateur integro-différentiel singulier Λ sur la droite réelle. Nous construisons une paire de transformations intégrales qui transmutent Λ et son dual $\widetilde{\Lambda}$ en l'opérateur d/dx. En utilisant les propriétés de ces opérateurs de transmutation, on définit une nouvelle analyse harmonique sur $\mathbb{R}$ correspondant à l'opérateur Λ.

1. Notations

We denote by $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. Recall that each function f in $\mathcal{E}(\mathbb{R})$ may be decomposed uniquely into the sum $f = f_e + f_o$, where the even part f_e is defined by $f_e(x) = (f(x) + f(-x))/2$ and the odd part f_o by $f_o(x) = (f(x) - f(-x))/2$. $\mathcal{E}_e(\mathbb{R})$ (resp. $\mathcal{E}_o(\mathbb{R})$) stands for the subspace of $\mathcal{E}(\mathbb{R})$ consisting of even (resp. odd) functions. For $a > 0$, $\mathcal{D}_a(\mathbb{R})$ designates the space of C^∞ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$. Put $\mathcal{D}(\mathbb{R}) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology. $\mathcal{D}_e(\mathbb{R})$ (resp. $\mathcal{D}_o(\mathbb{R})$) denotes the subspace of $\mathcal{D}(\mathbb{R})$ consisting of even (resp. odd) functions. For $a > 0$, let $\mathbf{H}_a$ be the space of entire, rapidly decreasing functions of exponential type a. Put $\mathbf{H} = \bigcup_{a>0} \mathbf{H}_a$, endowed with the inductive limit topology. Let $\mathcal{I}$ (resp. $\mathcal{J}$) denotes the map defined on $\mathcal{E}_e(\mathbb{R})$ (resp. $\mathcal{D}_o(\mathbb{R})$) by $\mathcal{I}h(x) = \frac{1}{A(x)} \int_0^x h(t)A(t)dt$ (resp. $\mathcal{J}h(x) = \int_{-\infty}^x h(t)dt$).

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2. Transmutation operators

In [4] we have considered the first-order singular differential-difference operator

$$\Lambda_0 f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right),$$

where

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

B being a positive C^∞ even function on $\mathbb{R}$. We have exploited a pair of transmutation operators between Λ_0 and the first derivative operator $\mathrm{d}/\mathrm{d}x$, to initiate a quite new harmonic analysis on the real line tied to Λ_0, in which several analytic structures on $\mathbb{R}$ were generalized. The key role in our investigation was played by the second-order differential operator

$$\Delta_0 f(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}f}{\mathrm{d}x},$$

which is linked to Λ_0 via the relationship

$$\Lambda_0^2 f = \Delta_0 f, \quad \text{for all } f \in \mathcal{E}_{\mathrm{e}}(\mathbb{R}).$$

Put

$$\Delta = \Delta_0 + q,$$

where q is a real-valued C^∞ even function on $\mathbb{R}$. The motivation of the present paper was to look for an integro-differential operator of the form

$$\Lambda = \Lambda_0 + M(x)\int_{-x}^{x} f(t)N(t)\mathrm{d}t$$

(M and N being two even functions) such that

$$\Lambda^2 f = \Delta f, \quad \text{for all } f \in \mathcal{E}_{\mathrm{e}}(\mathbb{R}). \tag{1}$$

A straightforward calculation shows that (1) is equivalent to

$$(2MN - q)f + \frac{2}{A}(AM)'\int_0^x fN\mathrm{d}t = 0,$$

for all $f \in \mathcal{E}_{\mathrm{e}}(\mathbb{R})$. The easiest choice was

$$AM = 1 \quad \text{and} \quad 2MN - q = 0,$$

that is,

$$\Lambda = \Lambda_0 + \frac{1}{A(x)}\int_0^x \left(\frac{f(t)+f(-t)}{2}\right)q(t)A(t)\mathrm{d}t.$$

The objective of this work is to establish for Λ results similar to those obtained for Λ_0 in [4]. This objective is achieved by using the crucial identity (1) and some basic facts about the differential operator Δ. Recall that Lions [2] has constructed an automorphism $\mathcal{X}$ of $\mathcal{E}_{\mathrm{e}}(\mathbb{R})$ satisfying

$$\mathcal{X}\frac{\mathrm{d}^2}{\mathrm{d}x^2}f = \Delta\mathcal{X}f \quad \text{and} \quad \mathcal{X}f(0) = f(0) \quad \text{for all } f \in \mathcal{E}_{\mathrm{e}}(\mathbb{R}).$$

The construction of the Lions operator $\mathcal{X}$ was aimed at allowing the resolution of certain mixed value problems. Later, Trimèche [5] has obtained for the Lions operator $\mathcal{X}$ the following integral representation:

$$\mathcal{X}f(x) = \int_0^{|x|}\mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \neq 0,\ f \in \mathcal{E}_{\mathrm{e}}(\mathbb{R}), \tag{2}$$