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Numerical analysis

Error estimates for stabilized finite element methods applied to ill-posed problems

Estimations d'erreurs pour des méthodes d'éléments finis stabilisées appliquées à des problèmes mal posés

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A R T I C L E I N F O

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A B S T R A C T

We propose an analysis for the stabilized finite element methods proposed in Burman (2013) [2] valid in the case of ill-posed problems for which only weak continuous dependence can be assumed. A priori and a posteriori error estimates are obtained without assuming coercivity or inf–sup stability of the continuous problem.

R É S U M É

Dans cette note, nous proposons une nouvelle analyse pour les méthodes d'éléments finis stabilisées introduites dans Burman (2013) [2], appliquées a des problèmes mal posés avec des propriétés de dépendance continue faibles. Nous obtenons des estimations a priori et a posteriori sans supposer ni coercitivité ni stabilité inf–sup de la forme bilinéaire du problème continu.

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1. Introduction

We are interested in the numerical approximation of ill-posed problems. The abstract theory will be illustrated by the following linear elliptic Cauchy problem. Let Ω be a convex polygonal (polyhedral) domain in $\mathbb{R}^d$, $d = 2, 3$, and consider the equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0 \quad \text{and} \quad \nabla u \cdot n = \psi & \text{on } \Gamma \end{cases} \tag{1}$$

where $\Gamma \subset \partial\Omega$ denotes a simply connected part of the boundary and $f \in L^2(\Omega)$, $\psi \in H^{\frac{1}{2}}(\Gamma)$. Introducing the spaces $V := \{v \in H^1(\Omega) : v|_\Gamma = 0\}$ and $W := \{v \in H^1(\Omega) : v|_{\Gamma'} = 0\}$, where $\Gamma' := \partial\Omega \setminus \Gamma$ and the forms $a(u, w) := \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x$, and $l(w) := \int_\Omega f w \, \mathrm{d}x + \int_\Gamma \psi w \, \mathrm{d}s$ Eq. (1) may be cast in the abstract weak formulation, find $u \in V$ such that

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$$a(u,w) = l(w) \quad \forall w \in W. \tag{2}$$

It is well known that the Cauchy problem (1) is not well-posed in the sense of Hadamard. If $l(w)$ is such that a sufficiently smooth, exact solution exists, conditional continuous dependence estimates can nevertheless be obtained [1].

The objective of the present paper is to study numerical methods for ill-posed problems of the form (2), where $a : V \times W \mapsto \mathbb{R}$ and $l : W \mapsto \mathbb{R}$ are a bilinear and a linear form. Assume that the linear form $l(w)$ is such that the problem (2) admits a unique solution $u \in V$. Define the following dual norm on l, $\|l\|_{W'} := \sup_{\substack{w\in W\\ \|w\|_W=1}} |l(w)|$. *Observe that we do not assume that* (2) *admits a unique solution for all* $l(w)$ *such that* $\|l\|_{W'} < \infty$. The stability property we assume to be satisfied by (2) is the following continuous dependence.

Assumption: continuous dependence on data. Consider the functional $j : V \mapsto \mathbb{R}$. Let $\Xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous, monotone increasing function with $\lim_{x\to 0^+} \Xi(x) = 0$. Let $\epsilon > 0$.

$$\boxed{\text{Assume that there holds } \|l\|_{W'} \le \epsilon \text{ in (2) then, for } \epsilon \text{ sufficiently small, } \big|j(u)\big| \le \Xi(\epsilon).} \tag{3}$$

For the example of the Cauchy problem (1), it is known [1, Theorems 1.7 and 1.9] that if (1) admits a unique solution $u \in H^1(\Omega)$, a continuous dependence of the form (3), with $0 < \epsilon < 1$, holds for

$$j(u) := \|u\|_{L^2(\omega)}, \omega \subset \Omega : \mathrm{dist}(\omega, \partial\Omega) =: d_{\omega,\partial\Omega} > 0 \quad \text{with } \Xi(x) := C_{u\varsigma} x^{\varsigma}, C_{u\varsigma} > 0, \varsigma := \varsigma(d_{\omega,\partial\Omega}) \in (0,1) \tag{4}$$

and for

$$j(u) := \|u\|_{L^2(\Omega)} \quad \text{with } \Xi(x) := C_u\big(\big|\log(x)\big| + C\big)^{-\varsigma} \text{ with } C_u, C > 0, \varsigma \in (0,1). \tag{5}$$

Note that to derive these results, $l(\cdot)$ is first associated with its Riesz representant in W (cf. [1, Eq. (1.31)] and discussion). The constant $C_{u\varsigma}$ in (4) grows monotonically in $\|u\|_{L^2(\Omega)}$ and C_u in (5) grows monotonically in $\|u\|_{H^1(\Omega)}$.

2. Finite element discretization

Let $\mathcal{K}_h$ be a shape regular, conforming, subdivision of Ω into non-overlapping triangles κ. The family of meshes $\{\mathcal{K}_h\}_h$ is indexed by the mesh parameter $h := \max(\mathrm{diam}(\kappa))$. Let $\mathcal{F}_I$ be the set of interior faces in $\mathcal{K}_h$ and $\mathcal{F}_\Gamma, \mathcal{F}_{\Gamma'}$ the set of element faces of $\mathcal{K}_h$ whose interior intersects Γ and Γ' respectively. We assume that the mesh matches the boundary of Γ so that $\mathcal{F}_\Gamma \cap \mathcal{F}_{\Gamma'} = \emptyset$. Let X_h^1 denote the standard finite element space of continuous, affine functions. Define $V_h := V \cap X_h^1$ and $W_h := W \cap X_h^1$. We may then write the finite element method: find $(u_h, z_h) \in V_h \times W_h$ such that,

$$\left.\begin{aligned} a(u_h, w_h) - s_W(z_h, w_h) &= l(w_h) \\ a(v_h, z_h) + s_V(u_h, v_h) &= s_V(u, v_h) \end{aligned}\right\} \quad \text{for all } (v_h, w_h) \in V_h \times W_h. \tag{6}$$

A possible choice of stabilization operators for the problem (1) are

$$s_V(u_h, v_h) := \sum_{F\in\mathcal{F}_I\cup\mathcal{F}_\Gamma} \int_F h_F[\partial_n u_h][\partial_n v_h]\,\mathrm{d}s, \quad \text{with } h_F := \mathrm{diam}(F) \tag{7}$$

and

$$s_W(z_h, w_h) := a(z_h, w_h) \quad \text{or} \quad s_W(z_h, w_h) := \sum_{F\in\mathcal{F}_I\cup\mathcal{F}_{\Gamma'}} \int_F h_F[\partial_n z_h][\partial_n w_h]\,\mathrm{d}s \tag{8}$$

where $[\partial_n u_h]$ denotes the jump of $\nabla u_h \cdot n_F$ for $F \in \mathcal{F}_I$ and when $F \in \mathcal{F}_\Gamma$ or $F \in \mathcal{F}_{\Gamma'}$ define $[\partial_n u_h]|_F := \nabla u_h \cdot n_{\partial\Omega}$. Unique existence of (u_h, z_h) solution to (6)–(8) follows using the arguments of [2, Proposition 3.3]. By inspection we have that the system (6) is consistent with (2) for $z_h = 0$. Taking the difference of (6) and the relation (2), with $w = w_h$, we obtain the Galerkin orthogonality,

$$a(u_h - u, w_h) - s_W(z_h, w_h) + a(v_h, z_h) + s_V(u_h - u, v_h) = 0 \quad \text{for all } (v_h, w_h) \in V_h \times W_h. \tag{9}$$

3. Hypotheses on forms and interpolants

Consider the general, positive semi-definite, symmetric stabilization operators, $s_V : V_h \times V_h \mapsto \mathbb{R}$, $s_W : W_h \times W_h \mapsto \mathbb{R}$. We assume that $s_V(u, v_h)$, with u the solution of (2), is explicitly known; it may depend on data from $l(w)$ or measurements of u. Assume that both s_V and s_W define semi-norms on $H^s(\Omega) + V_h$ and $H^s(\Omega) + W_h$ respectively, for some $s \ge 1$,

$$|v + v_h|_{s_Z} := s_Z(v + v_h, v + v_h)^{\frac{1}{2}}, \quad \forall v \in H^s(\Omega), v_h \in Z_h, \text{ with } Z = V, W. \tag{10}$$

Then assume that there exist interpolation operators $i_V : V \mapsto V_h$ and $i_W : W \mapsto W_h$ and norms $\|\cdot\|_{*,V}$ and $\|\cdot\|_{*,W}$ defined on V and W respectively, such that the form $a(u, v)$ satisfies the continuities