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A certain weighted variant of the embedding inequalities

Une variante avec poids des inégalités d'injection

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ABSTRACT

In this Note, for vector functions defined on unbounded domains of $\mathbb{R}^3$, we consider continuous embeddings of weighted homogeneous Sobolev spaces into weighted Lebesgue spaces. Sufficient conditions on power-type weights for the validity of the inequalities are investigated. Moreover, the related properties of the suitable approximation by smooth functions with a bounded support can be proved.

RÉSUMÉ

Dans cette Note, pour des fonctions vectorielles définies sur des domaines non bornés de $\mathbb{R}^3$, nous considérons des inégalités d'injection d'espaces de Sobolev homogènes avec poids dans des espaces de Lebesgue avec poids. Des conditions suffisantes pour justifier ces inégalités sont établies dans le cas de poids de type puissance. En outre, nous vérifions les propriétés d'approximation par des fonctions indéfiniment différentiables à support borné.

1. Introduction and formulation of the main results

The homogeneous Sobolev spaces of vector functions $\mathbf{D}_w^{1,q}(\Omega)$ are appropriate for the analysis of systems of partial differential equations and boundary-value problems in unbounded exterior domains Ω of $\mathbb{R}^3$, like the complementary set of one or more compact sets Ω^c in $\mathbb{R}^3$. The control of a suitable behavior at large distances is required for the solution vector fields. So a fundamental role in our treatment is played by the choice of admissible radial weights w in the q-class of Muckenhoupt weights.

We are inspired by Galdi's presentation of Sobolev classical embedding inequalities (see his book [3], Chapter II, Section 5) to provide the weighted embedding inequalities. Another approach by using full Sobolev spaces with radial weights can be found in the works of Amrouche, Girault and their collaborators (see, e.g., [2]); a generalization of Lemma II.5.2 of [3] in this functional setting is given by Alliot [1], see Proposition 3.8. Let us mention that there are several results on weighted full Sobolev spaces and embeddings, or even weighted embedding of homogeneous Sobolev spaces but with different weights (see [7,4,8,10]).

The following conditions $(A_1^\alpha)_q$ and $(A_2^\alpha)_q$ are preparatory and adapted to our analysis:

$$(A_1^\alpha)_q \quad \left(\int_R^r \frac{d\rho}{\rho^{\frac{2}{q-1}} w(\rho)^{\frac{1}{q-1}}}\right)^{q-1} \leqslant \begin{cases} c(q,\kappa)\cdot R^{-\alpha}, \text{ for some } \alpha>0, & \text{for } 1<q<3, \\ c(q,\kappa), & \text{for } q=3, \\ c(q,\kappa)\cdot r^{\alpha}, \text{ for some } \alpha>0, & \text{for } q>3 \end{cases}$$

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$$(A_2^\alpha)_q \quad \begin{cases} |\cdot|^{2-q-\alpha}(\ln|\cdot|)^{-q} \in \mathbf{L}^1_w(\Omega), & 1<q<3, \\ |\cdot|^{2-q+\alpha}(\ln|\cdot|)^{-q} \in \mathbf{L}^1_w(\Omega), & q \geqslant 3. \end{cases}$$

The conditions (A^α) we introduce above do not impose serious restriction on radial weights in the q-class of Muckenhoupt weights. For instance, when the weight is assumed to be a power-type function $w_\kappa(|\mathbf{x}|) := (1+|\mathbf{x}|)^\kappa$ for some $\kappa > 0$, the condition $(A_1^\alpha)_{1<q<3}$ is always true for $\alpha = \frac{3-q+\kappa}{q-1}$.

Let us fix some notations: $B_R(\mathbf{x}_0)$ means the $\mathbf{x}_0$-centered ball of radius R; we now set $\Omega^R(\mathbf{x}_0) := \Omega \setminus B_R(\mathbf{x}_0)$, $\Omega_R(\mathbf{x}_0) := \Omega \cap B_R(\mathbf{x}_0)$, and $\Omega_{R,r}(\mathbf{x}_0) := \Omega_r(\mathbf{x}_0) \setminus \Omega_R(\mathbf{x}_0)$ for a spherical shell. For any $\mathbf{x}_0 \in \mathbb{R}^3$, the value of $R > 0$ is assumed to be sufficiently large, more precisely, all used parameters $R > 0$ will have the following property (A_R):

$$(A_R) \quad \Omega^c \subset B_R(\mathbf{x}_0), \quad 0 < \delta(\Omega^c) < R.$$

Parameter R in the condition $(A_1^\alpha)_q$ is assumed to be sufficiently large in this sense.

Our objective is to establish the following results, where we assume concrete radial weights of the form w_κ:

Theorem 1 *(On a weighted embedding inequality). Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Assume that $\mathbf{u}$ is given in $\mathbf{D}^{1,q}_w(\Omega)$, $1<q<3$, with the weight $w = w_\kappa$ and $\kappa < \frac{3-q}{2q}$. Let the constant vector $\mathbf{u}_0$ be defined in Lemma 1.*

Then $(\mathbf{u}(\cdot) - \mathbf{u}_0)(|\cdot - \mathbf{x}_0|^{-1}) \in \mathbf{L}^q_w(\Omega^R(\mathbf{x}_0))$ for any $\mathbf{x}_0 \in \mathbb{R}^3$, $R > 0$ satisfying the condition (A_R). Moreover, there exists $K_1 = K_1(q, \mathbf{x}_0) > 0$ such that:

$$\left(\int_{\Omega^R(\mathbf{x}_0)} \left| \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}_0}{\mathbf{x} - \mathbf{x}_0} \right|^q w(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} \right)^{1/q} \leqslant K_1 |\mathbf{u} - \mathbf{u}_0|_{1,q,\Omega^R(\mathbf{x}_0);w}. \tag{1}$$

If Ω is locally Lipschitzian, denoting by $s(q) = \frac{3q}{3-q}$ the Sobolev exponent, there exists $K_2 = K_2(q) > 0$ such that:

$$\|\mathbf{u} - \mathbf{u}_0\|_{s(q),\Omega;w} \leqslant K_2 |\mathbf{u}|_{1,q,\Omega;w}. \tag{2}$$

Theorem 2 *(Another form of weighted embedding inequality). Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Assume that $\mathbf{u}$ is given in $\mathbf{D}^{1,q}_w(\Omega) \cap \mathbf{L}^q_{|\nabla w|}(\Omega)$, $1<q<3$, with the weight $w = w_\kappa$ and $\kappa < \frac{3-q}{q}$. Let the constant vector $\mathbf{u}_0$ be defined in Lemma 1.*

Then $(\mathbf{u}(\cdot) - \mathbf{u}_0)(|\cdot - \mathbf{x}_0|^{-1}) \in \mathbf{L}^q_w(\Omega^R(\mathbf{x}_0))$ for any $\mathbf{x}_0 \in \mathbb{R}^3$, $R > 0$ satisfying the condition (A_R). Moreover, there exists $K_3 = K_3(q, \mathbf{x}_0) > 0$ such that:

$$\left(\int_{\Omega^R(\mathbf{x}_0)} \left| \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}_0}{\mathbf{x} - \mathbf{x}_0} \right|^q w(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} \right)^{1/q} \leqslant K_3 \big(|\mathbf{u} - \mathbf{u}_0|_{1,q,\Omega^R(\mathbf{x}_0);w} + \|\mathbf{u} - \mathbf{u}_0\|_{q,\Omega^R(\mathbf{x}_0);|\nabla w|} \big). \tag{3}$$

If Ω is locally Lipschitzian, denoting by $s(q)$ the same value as in Theorem 1, there exists $K_4 = K_4(q) > 0$ such that:

$$\|\mathbf{u} - \mathbf{u}_0\|_{s(q),\Omega;w} \leqslant K_4 \big(|\mathbf{u}|_{1,q,\Omega;w} + \|\mathbf{u} - \mathbf{u}_0\|_{q,\Omega;|\nabla w|} \big). \tag{4}$$

Theorem 3 *(On the approximation by smooth functions, $1 \leqslant q < 3$). Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitzian exterior domain, $\mathbf{u} \in \mathbf{D}^{1,q}_w(\Omega)$, $1 \leqslant q < 3$, where the weight $w = w_\kappa$ satisfies the conditions $(A_1^\alpha)_{1<q<3}$ and $(A_2^\alpha)_{1<q<3}$. Let $\mathbf{u}_0$ be the constant vector given by Lemma 1.*

Then $\mathbf{u}$ can be approximated in the semi-norm $|\cdot|_{1,q,\Omega;w}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}$ has zero trace on the boundary $\partial\Omega$ and $\mathbf{u}_0 = \mathbf{0}$.

Corollary 1 *(The unweighted case, $1 \leqslant q < 3$). Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitzian exterior domain. The unconditional version of Lemma 1 where $w \equiv 1$ and $\alpha = \frac{3-q}{q-1}$ gives the constant vector $\mathbf{u}_0$.*

Then functions $\mathbf{u} \in \mathbf{D}^{1,q}(\Omega)$, $1 \leqslant q < 3$, can be approximated in the semi-norm $|\cdot|_{1,q,\Omega;1}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}$ has zero trace on the boundary $\partial\Omega$ and $\mathbf{u}_0 = \mathbf{0}$.

Remark 1. The corollary just shown improves the corresponding theorem in [3, Theorem II.7.1], indeed that properties $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ and $\mathbf{u}_0 = \mathbf{0}$ are not only sufficient but also necessary for approximating functions from $\mathbf{D}^{1,q}(\Omega)$ by smooth functions with compact support. As it is explained in [3], one can also replace the zero trace $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ by the condition $\psi\mathbf{u} \in \mathbf{W}^{1,q}_0(\Omega)$ for all $\psi \in C_0^\infty(\mathbb{R}^3)$ without assuming any regularity on $\partial\Omega^c$.

Theorem 4 *(On the approximation by smooth functions, $q \geqslant 3$). Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitzian exterior domain, $\mathbf{u} \in \mathbf{D}^{1,q}_w(\Omega)$, $q \geqslant 3$ where the weight $w = w_\kappa$ satisfies the conditions $(A_1^\alpha)_{q\geqslant 3}$ and $(A_2^\alpha)_{q\geqslant 3}$.*

Then $\mathbf{u}$ can be approximated in the semi-norm $\|\cdot\|_{1,q,\Omega;w}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$.