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Partial differential equations

Effective stability for slow time-dependent near-integrable Hamiltonians and application

Stabilité effective pour des hamiltoniens presque intégrables lentement non autonomes et application

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ABSTRACT

The aim of this note is to prove a result of effective stability for a non-autonomous perturbation of an integrable Hamiltonian system, provided that the perturbation depends slowly on time. Then we use this result to clarify and extend a stability result of Giorgilli and Zehnder for a mechanical system with an arbitrary time-dependent potential.

RÉSUMÉ

Le but de cette note est de démontrer un résultat de stabilité effective pour une perturbation non autonome d'un système hamiltonien intégrable, sous la condition que la perturbation dépende lentement du temps. Nous utilisons ensuite ce résultat pour clarifier et généraliser un résultat de stabilité de Giorgilli et Zehnder pour des systèmes mécaniques dont le potentiel dépend arbitrairement du temps.

1. Introduction

Let $n \in \mathbb{N}$, $n \geqslant 2$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and consider the Hamiltonian system defined by $H : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$H(\theta, I, t) = h(I) + \varepsilon f(\theta, I, t), \quad (\theta, I, t) = (\theta_1, \ldots, \theta_n, I_1, \ldots, I_n, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}, \ \varepsilon > 0. \tag{1}$$

Nekhoroshev proved [11] that, whenever h is steep (see Section 2 for a definition), $f(\theta, I, t) = f(\theta, I)$ is time-independent and H is real-analytic, there exist positive constants $\varepsilon_0, c_1, c_2, c_3, a, b$ such that for all $\varepsilon \leqslant \varepsilon_0$ and all solutions $(\theta(t), I(t))$, if $|t| \leqslant c_2 \exp(c_3 \varepsilon^{-a})$, then we have the following stability estimate:

$$\left|I(t) - I(0)\right| = \max_{1 \leqslant i \leqslant n} \left|I_i(t) - I_i(0)\right| \leqslant c_1 \varepsilon^b. \tag{2}$$

In the particular case where h is (strictly uniformly) convex or quasi-convex, following a work of Lochak [7] it was proved [9,13], using preservation of energy arguments, that one can choose $a = b = (2n)^{-1}$ in (2), and that these values are close to optimal (in the general steep case, however, there are still no realistic values for these stability exponents a and b).

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The purpose of this note is to discuss to which extent a stability estimate similar to (2) holds true if the perturbation is allowed to depend on time.

Assume first that f depends periodically on time, that is $f(\theta, I, t) = f(\theta, I, t+T)$ in (1) for some $T > 0$ (we may assume $T = 1$ by a time scaling). Removing the time dependence by adding an extra degree of freedom, the Hamiltonian is equivalent to:

$$\tilde{H}(\theta, \varphi, I, J) = \tilde{h}(I, J) + \varepsilon f(\theta, \varphi, I), \quad (\theta, \varphi = t, I, J) \in \mathbb{T}^n \times \mathbb{T} \times \mathbb{R}^n \times \mathbb{R},\ \tilde{h}(I, J) = h(I) + J.$$

It turns out that if h is convex, then $\tilde{h}$ is quasi-convex and so (2) holds true with $a = b = (2(n+1))^{-1}$. In general, it is possible for $\tilde{h}$ to be steep, in which case (2) is satisfied, but it is not clear how to formulate a condition on h (and not on $\tilde{h}$) to ensure that (2) holds true.

Now assume that f depends quasi-periodically on time, that is $f(\theta, I, t) = F(\theta, I, t\omega)$ in (1) for some function $F : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ and some vector $\omega \in \mathbb{R}^m$, which we can assume to be non-resonant ($k \cdot \omega \neq 0$ for any non-zero $k \in \mathbb{Z}^m$). As before, the time dependence can be removed by adding m degrees of freedom and we are led to consider $\tilde{H}(\theta, \varphi, I, J) = \tilde{h}(I, J) + \varepsilon f(\theta, \varphi, I)$, but this time:

$$(\theta, \varphi = t\omega, I, J) \in \mathbb{T}^n \times \mathbb{T}^m \times \mathbb{R}^n \times \mathbb{R}^m, \qquad \tilde{h}(I, J) = h(I) + \omega \cdot J.$$

It was conjectured by Chirikov [3], and then again by Lochak [8], that if h is convex and ω satisfies a Diophantine condition of exponent $\tau \geqslant m - 1$ (there exists a constant $\gamma > 0$ such that $|k \cdot \omega| \geqslant \gamma |k|^{-\tau}$ for any non-zero $k \in \mathbb{Z}^m$), then the estimate (2) holds true and, moreover, we can choose $a = b = (2(n+1+\tau))^{-1}$. If $m = 1$, then $\tau = 0$ and we are in the periodic case, so the conjecture is true. However, if $m > 1$, $\tilde{h}$ cannot be steep and the problem is still completely open. Even though the conjecture is sometimes considered as granted (without the explicit values for a and b, see, for instance, [6]), there is still no proof. Needless to say that the situation in the general case (without the convexity assumption on h) is even more complicated.

In a different direction, Giorgilli and Zehnder [4] considered the following time-dependent Hamiltonian:

$$G(\theta, I, t) = h_2(I) + V(\theta, t), \quad (\theta, I, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R},\ h_2(I) = I_1^2 + \cdots + I_n^2,$$

and proved the following Nekhoroshev-type result: if G is real-analytic and V is uniformly bounded, then for R sufficiently large, if I_0 belongs to the open ball B_R of radius R centered at the origin, then $I(t) \in B_{2R}$ for $|t| \leqslant c_2 \exp(c_3 R^d)$ for some positive constants c_2, c_3, and d. Even though such a system is clearly not of the form (1), the fact that no restriction on the time dependence is imposed in their result has led to several confusions. In [4], the authors themselves assert that “extra work is needed because the time dependence is not assumed to be periodic or quasi-periodic”. Even more surprising, one can read (in [10] for instance) that this result implies that the estimate (2) holds true for (1) without any restriction on the time dependence. Concerning the latter assertion, it is simply wrong and it seems very unlikely to have a non-trivial stability estimate for (1) with an arbitrary time dependence. As for the former assertion, it is not difficult to see that the system considered in [4] can be given the form (1), but with a perturbation depending “slowly” (and not arbitrarily) on time (see Section 2 for a definition of what we mean by “slowly” depending on time, and Section 3 for more details on the system considered in [4]). We will show in Section 2 that for a Hamiltonian system depending “slowly” on time, essentially classical techniques can be used to prove that (2) holds true, and that the non-periodicity or non-quasi-periodicity of time in this restricted context plays absolutely no role (as a matter of fact, we already explained that, for a periodic or quasi-periodic time dependence that is not slow, basic questions are still open). Then, in Section 3, we will use this result to derive, in a simpler way, a more general statement than the one contained in [4].

2. A stability result

For a given $\rho > 0$, recall that B_ρ is the open ball in $\mathbb{R}^n$ of radius ρ (with respect to the supremum norm) around the origin. A function $h \in C^2(B_\rho)$ is said to be steep if, for any affine subspace S of $\mathbb{R}^n$ intersecting B_ρ, the restriction $h_{|S}$ has only isolated critical points (it is not the original definition of Nekhoroshev, but it is equivalent to it, see [5] and [12]). We will assume that the operator norm $|\nabla^2 h(I)|$ is bounded uniformly in $I \in B_\rho$. Then, given $r, s > 0$, let us define the complex domain:

$$\mathcal{D}_{r,s} = \left\{(\theta, I, t) \in \left(\mathbb{C}^n/\mathbb{Z}^n\right) \times \mathbb{C}^n \times \mathbb{C} \;\middle|\; \left|\left(\mathrm{Im}(\theta_1), \ldots, \mathrm{Im}(\theta_n)\right)\right| < s,\ \left|\mathrm{Im}(t)\right| < s,\ d(I, B_\rho) < r\right\},$$

where the distance d is induced by the supremum norm. For a fixed constant $\lambda > 0$ and a “small” parameter $0 < \varepsilon \leqslant 1$, we consider $H(\theta, I, t) = h(I) + \varepsilon f(\theta, I, \varepsilon^\lambda t)$ defined on $\mathcal{D}_{r,s}$, real-analytic (that is H is analytic and real-valued for real arguments), and we assume that $|f(\theta, I, t)| \leqslant 1$ for any $(\theta, I, t) \in \mathcal{D}_{r,s}$.

Theorem 2.1. *Under the previous assumptions, there exist positive constants $\varepsilon_0, c_1, c_2, c_3$, that depend on $n, \rho, h, r, s, \lambda$, and positive constants a, b that depend only on n, h, such that if $\varepsilon \leqslant \varepsilon_0$, for all solutions $(\theta(t), I(t))$ of the Hamiltonian system defined by H, if $I(0) \in B_{\rho/2}$, then the estimate $|I(t) - I(0)| \leqslant c_1 \varepsilon^b$ holds true for all time $|t| \leqslant c_2 \exp(c_3 \varepsilon^{-a})$.*