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Automation (theoretical)

Growth bound of delay-differential algebraic equations

Taux de croissance des équations algébro-différentielles à retards

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A R T I C L E   I N F O

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A B S T R A C T

This paper deals with delay-differential algebraic equations, a large class of linear and finite-memory functional differential equations. We introduce several representations of delay operators that provide a simple definition for the concept of solutions of such systems. Then we study exponential solutions and prove that the rightmost zeros of a system characteristic function determine its growth bound.

R É S U M É

Cet article traite des équations algébro-différentielles à retards, un large sous-ensemble des équations différentielles fonctionnelles linéaires à mémoire finie. Nous introduisons différentes représentations des opérateurs de retard, qui fournissent une définition simple du concept de solution de tels systèmes. Ensuite, nous étudions les solutions exponentielles et prouvons que les zéros les plus à droite de la fonction caractéristique d'un système déterminent son taux de croissance.

1. Introduction

Delay-differential algebraic equations (DDAE) are a class of functional differential equations (FDE) whose variables are connected through integrators and finite-memory delay operators. Such a system of equations, with variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a finite memory length r, has the structure:

$$\left| \begin{aligned} \dot{x}(t) &= Ax_t + By_t \\ y(t) &= Cx_t + Dy_t \end{aligned} \right. \tag{1}$$

where z_t refers to the memory of the variable z at time t:

$$\operatorname{dom} z_t = [-r, 0] \quad \text{and} \quad \forall \theta \in [-r, 0],\ z_t(\theta) = z(t + \theta) \tag{2}$$

and the symbols A, B, C, D denote delay operators:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : C\big([-r, 0], \mathbb{C}^{n+m}\big) \to \mathbb{C}^{n+m}, \quad \text{linear and bounded.} \tag{3}$$

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This class of systems has been already considered in [1,2,4,10–12], but most of the literature has been focused on some of its subclasses: equations of retarded type, neutral type and difference equations. Nonetheless, the general model is important: it is required in the modelling of some physical phenomena such as lossless propagation (see [11] and the references therein) and in the control of dead-time systems when standard methods such as finite-spectrum assignment [9] are used.

A classic stability criterion determines the growth bound of a DDAE system from the location of the rightmost zeros of its characteristic function. The validity of this criterion has already been established with several methods under various restrictive assumptions: for systems of retarded type, for systems of neutral type, and difference equations whose difference operator combines only discrete and distributed delays [7] or satisfies a "jump" assumption [6], and for DDAE with discrete delays and stable difference operators [5].

We demonstrate in this paper that a single method, that combines the use of the Gearhart–Prüss theorem with bounds for the characteristic matrix inverse, established by complex analysis, can be used to prove this criterion in all these special cases. Actually, we require only the DDAE system to have a strictly causal difference operator, an assumption already used to ensure the well-posedness of the system. To the best knowledge of the author, this general result was not available.

Matrix-valued measures provide a concrete representation of delay operators: any linear bounded operator $L : C([-r,0],\mathbb{C}^j) \to \mathbb{C}^i$ corresponds to a unique countably additive function on the bounded Borel subsets of $\mathbb{R}$, supported on $[-r,0]$, with values in $\mathbb{C}^{i\times j}$. This alternate representation – that we still denote by L – is related to the initial operator by:

$$L\phi = \int \mathrm{d}L\,\phi := \sum_l \left[\int \phi_k\,\mathrm{d}L_{lk}\right] e_l \tag{4}$$

where $(e_1,\dots,e_i)$ denotes the canonical basis of $\mathbb{C}^i$. Let L^* be the measure obtained by symmetry around $t=0$ of L, such that for any bounded Borel set B, $L^*(B)=L(-B)$ and let $*$ be the convolution between time-dependent locally integrable functions – or more generally measures – of left-sided bounded support. The convolution of two scalar, vector or matrix-valued measures of compatible dimensions is defined as the combination of scalar convolution and linear algebra product; for example, for two matrix-valued measures A and B, $A*B$ is the matrix-valued measure such that $(A*B)_{ij} := \sum_k A_{ik}*B_{kj}$. We also implicitly extend functions defined on a subset of $\mathbb{R}$ by zero outside of their domain. With these conventions, for any continuous function z defined on $[-r,+\infty)$, we have $\forall t>0$, $Lz_t = (L^* * z)(t)$. As the right-hand side of this equation is still properly defined – as a locally integrable function of t – if z is merely locally integrable, we may rewrite Eq. (1) as a convolution equation.

Let e be the Heaviside function. We say that a pair of locally integrable functions (x,y), defined on $[-r,+\infty)$, with values in $\mathbb{C}^{n+m}$, is a (locally integrable) solution of (1) if there is an $f\in\mathbb{C}^n$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix}(t) = \begin{bmatrix} e*A^* & e*B^* \\ C^* & D^* \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}(t) + \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{for a.e. } t>0. \tag{5}$$

We assume in the sequel that the difference operator D is strictly causal, that is $D(\{0\})=0$. This condition ensures that this system of equations defines a well-posed initial value problem in the Hilbert product space $X=\mathbb{C}^n\times L^2([-r,0],\mathbb{C}^{n+m})$, see [1,2,12]: given any $(\phi,\chi,\psi)\in X$, there is a unique solution (x,y) such that $(x(0^+),x_0,y_0)=(\phi,\chi,\psi)$ and the mapping $(t\in\mathbb{R}_+\mapsto \exp(\mathcal{A}t))$ given by $(x(t^+),x_t,y_t)=\exp(\mathcal{A}t)(\phi,\chi,\psi)$ for $t\geqslant 0$ is a strongly continuous semigroup on X.

2. Exponential solutions – characteristic matrix and resolvent operator

We denote by Δ the characteristic matrix of system (1), defined at any point $s\in\mathbb{C}$ by

$$\Delta(s) = \begin{bmatrix} sI_n & 0 \\ 0 & I_m \end{bmatrix} - \mathscr{L}\begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}(s) \tag{6}$$

where $\mathscr{L}$ is the Laplace transform.

The determinant of the characteristic matrix – the characteristic function – and its adjugate both have a quasi-polynomial structure:

$$\det\Delta(s) = \sum_{i=0}^{n} c_i(s)s^i, \qquad \operatorname{adj}\Delta(s) = \sum_{i=0}^{n} C_i(s)s^i \tag{7}$$

where the c_i (resp. C_i) are entire functions (resp. matrices of entire functions) bounded on any right-hand plane. Moreover, the leading coefficient of the characteristic function is given by $c_n(s)=\det\Delta_0(s)$, where Δ_0 is the characteristic matrix of the system $y(t)=Dy_t$. Lemma 2.1 establishes elementary properties of $\det\Delta_0$ and Lemma 2.2 describes how the zeros of $\det\Delta_0$ and $\det\Delta$ are connected.

For any real number σ, we denote by P_σ the open half-plane $\{s\in\mathbb{C} \mid \Re(s)>\sigma\}$ and for any positive η, we denote by Z_η the set of complex numbers whose distance to the zeros of $\det\Delta_0$ it at most η:

$$Z_\eta = \left\{ s\in\mathbb{C} \;\middle|\; \exists z\in\mathbb{C},\ \det\Delta_0(z)=0 \wedge |s-z|\leqslant\eta \right\}. \tag{8}$$