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Harmonic Analysis

The John–Nirenberg inequality with sharp constants

Meilleures constantes dans l'inégalité de John–Nirenberg

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ABSTRACT

We consider the one-dimensional John–Nirenberg inequality:

$$\left|\left\{x \in I_0\colon \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| \leqslant C_1|I_0| \exp\left(-\frac{C_2}{\|f\|_*}\alpha\right).$$

A. Korenovskii found that the sharp C_2 here is $C_2 = 2/\mathrm{e}$. It is shown in this paper that if $C_2 = 2/\mathrm{e}$, then the best possible C_1 is $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$.

RÉSUMÉ

On considère l'inégalité de John–Nirenberg unidimensionnelle :

$$\left|\left\{x \in I_0\colon \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| \leqslant C_1|I_0| \exp\left(-\frac{C_2}{\|f\|_*}\alpha\right).$$

A. Korenovskii a montré que la meilleure constante C_2 était égale à $2/\mathrm{e}$. Dans cette Note, on montre que si $C_2 = 2/\mathrm{e}$, alors la meilleure constante possible pour C_1 est $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$.

1. Introduction

Let $I_0 \subset \mathbb{R}$ be an interval and let f be an integrable function on I_0. Given a measurable set $E \subset \mathbb{R}$, denote by $|E|$ its Lebesgue measure. Given a subinterval $I \subset I_0$, set $f_I = \frac{1}{|I|}\int_I f$ and

$$\Omega(f; I) = \frac{1}{|I|}\int_I \left|f(x) - f_I\right| \mathrm{d}x.$$

We say that $f \in BMO(I_0)$ if $\|f\|_* \equiv \sup_{I \subset I_0} \Omega(f; I) < \infty$. The classical John–Nirenberg inequality [1] says that there are $C_1, C_2 > 0$ such that for any $f \in BMO(I_0)$,

$$\left|\left\{x \in I_0\colon \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| \leqslant C_1|I_0| \exp\left(-\frac{C_2}{\|f\|_*}\alpha\right) \quad (\alpha > 0).$$

A. Korenovskii [4] (see also [5, p. 77]) found the best possible constant C_2 in this inequality, namely, he showed that $C_2 = 2/\mathrm{e}$:

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$$\left|\left\{x \in I_0 \colon \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| \leqslant C_1 |I_0| \exp\left(-\frac{2/\mathrm{e}}{\|f\|_*}\alpha\right) \quad (\alpha > 0), \tag{1.1}$$

and in general the constant $2/\mathrm{e}$ here cannot be increased.

A question about the sharp C_1 in (1.1) remained open. In [4], (1.1) was proved with $C_1 = \mathrm{e}^{1+2/\mathrm{e}} = 5.67323\ldots$. The method of the proof in [4] was based on the Riesz sunrise lemma and on the use of non-increasing rearrangements. In this paper, we give a different proof of (1.1), yielding the sharp constant $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}} = 2.17792\ldots$.

Theorem 1.1. *Inequality (1.1) holds with $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$, and this constant is the best possible.*

We also use as the main tool the Riesz sunrise lemma. But instead of the rearrangement inequalities, we obtain a direct pointwise estimate for any *BMO*-function (see Theorem 2.2 below). The proof of this result is inspired (and close in spirit) by a recent decomposition of an arbitrary measurable function in terms of mean oscillations (see [2,6]).

We mention several recent papers [7,8] where sharp constants in some different John–Nirenberg-type estimates were found by means of the Bellman function method.

2. Proof of Theorem 1.1

We shall use the following version of the Riesz sunrise lemma [3].

Lemma 2.1. *Let g be an integrable function on some interval $I_0 \subset \mathbb{R}$, and suppose $g_{I_0} \leqslant \alpha$. Then there is at most countable family of pairwise disjoint subintervals $I_j \subset I_0$ such that $g_{I_j} = \alpha$, and $g(x) \leqslant \alpha$ for almost all $x \in I_0 \setminus (\bigcup_j I_j)$.*

Observe that the family $\{I_j\}$ in Lemma 2.1 may be empty if $g(x) < \alpha$ a.e. on I_0.

Theorem 2.2. *Let $f \in BMO(I_0)$, and let $0 < \gamma < 1$. Then there is at most countable decreasing sequence of measurable sets $G_k \subset I_0$ such that $|G_k| \leqslant \min(2\gamma^k, 1)|I_0|$ and for a.e. $x \in I_0$,*

$$\left|f(x) - f_{I_0}\right| \leqslant \frac{\|f\|_*}{2\gamma}\sum_{k=0}^{\infty}\chi_{G_k}(x). \tag{2.1}$$

Proof. Given an interval $I \subseteq I_0$, set $E(I) = \{x \in I \colon f(x) > f_I\}$. Let us show that there is at most a countable family of pairwise disjoint subintervals $I_j \subset I_0$ such that $\sum_j |I_j| \leqslant \gamma|I_0|$ and for a.e. $x \in I_0$,

$$(f - f_{I_0})\chi_{E(I_0)} \leqslant \frac{\|f\|_*}{2\gamma}\chi_{E(I_0)} + \sum_j (f - f_{I_j})\chi_{E(I_j)}. \tag{2.2}$$

We apply Lemma 2.1 with $g = f - f_{I_0}$ and $\alpha = \frac{\|f\|_*}{2\gamma}$. One can assume that $\alpha > 0$ and the family of intervals $\{I_j\}$ from Lemma 2.1 is non-empty (since otherwise (2.2) holds trivially only with the first term on the right-hand side). Since $g_{I_j} = \alpha$, we obtain:

$$\sum_j |I_j| = \frac{1}{\alpha}\int_{\bigcup_j I_j}(f - f_{I_0})\,\mathrm{d}x \leqslant \frac{1}{\alpha}\int_{\{x\in I_0\colon f(x) > f_{I_0}\}}(f - f_{I_0})\,\mathrm{d}x$$
$$= \frac{1}{2\alpha}\Omega(f; I_0)|I_0| \leqslant \gamma|I_0|.$$

Since $g_{I_j} = \alpha$, we have $f_{I_j} = f_{I_0} + \alpha$, and hence:

$$f - f_{I_0} = (f - f_{I_0})\chi_{I_0\setminus\bigcup_j I_j} + \alpha\chi_{\bigcup_j I_j} + \sum_j (f - f_{I_j})\chi_{I_j}.$$

This proves (2.2) since $f - f_{I_0} \leqslant \alpha$ a.e. on $I_0 \setminus \bigcup_j I_j$.

The sum on the right-hand side of (2.2) consists of the terms of the same form as the left-hand side. Therefore, one can proceed iterating (2.2). Denote $I_j^1 = I_j$, and let I_j^k be the intervals obtained after the k-th step of the process. Iterating (2.2) m times yields:

$$(f - f_{I_0})\chi_{E(I_0)} \leqslant \frac{\|f\|_*}{2\gamma}\sum_{k=0}^{m}\sum_j \chi_{E(I_j^k)}(x) + \sum_i (f - f_{I_i^{m+1}})\chi_{E(I_i^{m+1})}$$