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Partial Differential Equations/Optimal Control

A Hamilton–Jacobi PDE in the space of measures and its associated compressible Euler equations

Une EDP de Hamilton–Jacobi dans l'espace des mesures et ses équations d'Euler compressibles associées

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ABSTRACT

We introduce a class of action integrals defined over probability-measure-valued path space. Minimal action exists in this context and gives weak solution to a compressible Euler equation. We prove that the Hamilton–Jacobi PDE associated with such variational formulation of Euler equation is well posed in viscosity solution sense.

RÉSUMÉ

Nous introduisons une classe d'intégrales d'action définies sur l'espace des chemins à valeurs mesures de probabilité. Dans ce contexte l'action minimale existe et donne une solution faible d'une équation d'Euler compressible. Nous montrons que l'équation de Hamilton Jacobi associ'ee à la formulation variationnelle de l'équation d'Euler est bien posée dans le sens des solutions de viscosité.

1. Introduction

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the space of Borel probability measures over $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} |x|^2 \rho(\mathrm{d}x) < \infty$ endowed with the Wasserstein 2-metric d. $AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ is the class of $\mathcal{P}_2(\mathbb{R}^d)$-valued absolute continuous curves. Each $\rho(\cdot)$ in such class satisfies the continuity equation $\dot\rho := \partial_t \rho = -\operatorname{div}(\rho u)$ for some u (Theorem 8.3.1 of [1]). This equation expresses a conservation of mass property and naturally introduces a class of parameterized curves, which motivates the following notion of tangent space and associated geometric structure on $\mathcal{P}_2(\mathbb{R}^d)$ (Chapter 8 of [1,6]):

$$H_{-1,\rho}(\mathbb{R}^d) := \{ m \in \mathcal{D}'(\mathbb{R}^d) \colon \|m\|_{-1,\rho} < \infty \}, \quad \|m\|^2_{-1,\rho} := \sup_{\varphi \in C_c^\infty(\mathbb{R}^d)} \{ 2\langle m, \varphi\rangle - \|\varphi\|^2_{1,\rho} \}. \tag{1}$$

In the above, $\|\varphi\|^2_{1,\rho} = \int_{\mathbb{R}^d} |\nabla\varphi|^2 \,\mathrm{d}\rho$. It follows that $\int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}\rho = \|\dot\rho\|^2_{-1,\rho}$. We denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition 1.1 *(Gradient of a function).* Let $f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \overline{\mathbb{R}}$, $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$, and $f(\rho_0)$ be finite. We say that gradient of f at ρ_0, denoted $\operatorname{grad} f(\rho_0)$, exists, if it can be identified as the unique element in $\mathcal{D}'(\mathbb{R}^d)$ satisfying the following property: for

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every $p \in C_c^\infty(\mathbb{R}^d)$ and the family of push forward of ρ_0 through the flow generated by ∇p, i.e. $\{\rho^p(t) \in \mathcal{P}_2(\mathbb{R}^d)\colon\ t \in \mathbb{R}\}$ with $\partial_t \rho^p + \operatorname{div}(\rho^p \nabla p) = 0$ and $\rho^p(0) = \rho_0$, we have $\lim_{t\to 0} t^{-1}(f(\rho^p(t)) - f(\rho^p(0))) =: \langle \operatorname{grad} f(\rho_0), p\rangle$.

Let $R(\rho\|\mu) := \int_{\mathbb{R}^d} \mathrm{d}\rho \log \frac{\mathrm{d}\rho}{\mathrm{d}\mu}$ denote relative entropy, define Gibbs measure $\mu^\Psi(\mathrm{d}x) := Z_\Psi^{-1} e^{-\Psi}$ with $Z_\Psi = \int_{\mathbb{R}^d} e^{-\Psi}\,\mathrm{d}x$, and entropy functional $S(\rho) := R(\rho\|\mu^\Psi)$. It follows then $\operatorname{grad} S(\rho) = -\Delta\rho - \operatorname{div}(\rho\nabla\Psi)$ whenever $S(\rho) < \infty$. Let $\psi := |\nabla\Psi|^2 - 2\Delta\Psi$, the Fisher information $I(\rho) := \|\operatorname{grad} S(\rho)\|_{-1,\rho}^2 = \int_{\mathbb{R}^d} \frac{|\nabla\rho|^2}{\rho}\,\mathrm{d}x + \int_{\mathbb{R}^d} \psi\,\mathrm{d}\rho$ (Appendix D.6 of [3]). Let $\nu > 0$, we introduce a modified kinetic energy $T(\rho,\dot\rho) := \frac12\|\dot\rho + \nu \operatorname{grad} S(\rho)\|_{-1,\rho}^2$ to reinforce entropy dissipation (see [4] and its appendix). Let potential energy

$$V(\rho) := \int_{\mathbb{R}^d} \phi(x)\rho(\mathrm{d}x) + \frac12 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi(x-y)\rho(\mathrm{d}x)\rho(\mathrm{d}y) + \int_{\mathbb{R}^d} F\big(\rho(x)\big)\,\mathrm{d}x.$$

Without pursuing generality, we assume that $\Phi, \phi \in C^1(\mathbb{R}^d)$ have sub-quadratic growth, $\Phi(-x) = \Phi(x)$, $\Psi \in C^4(\mathbb{R}^d)$ is quasi-convex and that the leading order terms for both Ψ and ψ have polynomial growth of order bigger than 2 (e.g. $\Psi(x) = \frac14|x|^4 - |x|^2$). Finally, let $F \in C^1$ be such that $|F(r)| \leqslant cr^\gamma$, $|rF'(r)| \leqslant c(1+r^\gamma)$ for some finite $c \geqslant 0$ and some $\gamma \geqslant 1$ where $\gamma \in [1, 1+\frac{2}{d})$ when $d \geqslant 3$ and $\gamma \in [1,2)$ when $d = 1,2$. For notational convenience, we set $V(\rho) = -\infty$ whenever $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ has no Lebesgue density. The following is a consequence of Sobolev inequality and the fact that $\int_{\mathbb{R}^d}\rho(\mathrm{d}x) = 1$. See [4]:

Lemma 1.2. *There exists a right continuous nondecreasing sub-linear function $\zeta : R_+ \mapsto \mathbb{R}_+$ with $|V(\rho)| \leqslant \zeta(I(\rho))$. Moreover, V is continuous on finite level sets of I.*

For $\rho(\cdot) \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ with $S(\rho(0)) < \infty$, by the calculus in [1], $\int_0^T T(\rho,\dot\rho)\,\mathrm{d}t = \frac12\int_0^T (\|\dot\rho\|_{-1,\rho}^2 + \nu^2 I(\rho))\,\mathrm{d}t + \nu(S(\rho(T)) - S(\rho(0)))$. This observation motivates us considering Lagrangians L and $\hat L : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{D}'(\mathbb{R}^d) \mapsto \mathbb{R}\cup\{+\infty\}$ by $L := T - V$ and $\hat L := \frac12\|\dot\rho\|_{-1,\rho}^2 + \frac{\nu^2}{2} I - V$, where $\hat L$ is understood as $+\infty$ when $V = +\infty$. $\hat L$ takes value in $\mathbb{R}\cup\{+\infty\}$. L, however, is only well defined when V is bounded from above in bounded sets of $\mathcal{P}_2(\mathbb{R}^d)$ (e.g. $F(r) \leqslant cr$ for some $c > 0$ will ensure this). Denote

$$A_T\big[\rho(\cdot)\big] := \int_0^T L(\rho,\dot\rho)\,\mathrm{d}t, \qquad J_T\big[\rho(\cdot)\big] := \int_0^T \hat L(\rho,\dot\rho)\,\mathrm{d}t, \quad \rho(\cdot) \in C\big([0,T];\mathcal{P}_2\big(\mathbb{R}^d\big)\big). \tag{2}$$

When both A_T and J_T are well defined and $S(\rho(0)) < \infty$, we have the following useful identity: $A_T[\rho(\cdot)] = J_T[\rho(\cdot)] + \nu(S(\rho(T)) - S(\rho(0)))$ for $\rho \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$. Action minimizer for A_T and J_T are the same under mild conditions, and solves a compressible Euler equation (Theorem 2.1)

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P(\rho) = -\rho\nabla(\phi + \Phi*\rho) - 2\nu^2\rho\nabla\left(\dfrac{\Delta\sqrt\rho}{\sqrt\rho} - \dfrac14\psi\right) \\ P(\rho) = \rho F'(\rho) - F(\rho). \end{cases} \tag{3}$$

If (ρ,u) are smooth for (3) to hold in classical sense, then it is also a weak solution as defined below.

Definition 1.3 *(Weak solution).* (ρ,u) is called a weak solution to system (3) if the following holds: $\rho(\cdot) \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ with $S(\rho(T)) + \int_0^T I(\rho(t))\,\mathrm{d}t < \infty$; $u : (0,T)\times\mathbb{R}^d \mapsto \mathbb{R}^d$ is Borel measurable satisfying $\int_0^T\int_{\mathbb{R}^d} |u(t,x)|^2\rho(t)\,\mathrm{d}t < \infty$; moreover, $\partial_t\rho + \operatorname{div}(\rho u) = 0$ holds in the distribution sense and

$$\int_0^T\int_{\mathbb{R}^d}\Big[u(t,x)\cdot\big(\partial_t\xi(t,x) + (u\cdot\nabla)\xi(t,x)\big)\rho(t,x) + P(\rho)\operatorname{div}\xi - \big(\nabla(\phi+\Phi*\rho)\cdot\xi\big)\rho(t,x)$$
$$+ \nu^2\left(-\frac{\nabla\rho}{\rho}\cdot D\xi\cdot\frac{\nabla\rho}{\rho} + \Delta\operatorname{div}\xi + \frac12\xi\cdot\nabla\psi\right)\rho(t,x)\Big]\,\mathrm{d}x\,\mathrm{d}t = 0,$$

holds for every $\xi \in C_c^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$, where $D\xi = (\partial_i\xi_j)_{(i,j)}$ is a matrix.

A satisfactory Hamilton–Jacobi PDE theory can also be developed (Theorem 2.2), based upon a Hamiltonian induced by the Lagrangian L, not the $\hat L$. For $V(\rho) < \infty$ and $n = -\operatorname{div}(\rho\nabla p)$ with $p \in C_c^\infty(\mathbb{R}^d)$, let

$$H(\rho,n) := \sup_{m\in H_{-1,\rho}(\mathbb{R}^d)} \big(\langle n,m\rangle_{-1,\rho} - L(\rho,m)\big) = -\big\langle \nu\operatorname{grad} S(\rho), n\big\rangle_{-1,\rho} + \frac12\|n\|_{-1,\rho}^2 + V(\rho).$$