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Numerical Analysis

A remark on supercloseness and extrapolation of the quadrilateral Han element for the Stokes equations

Une remarque concernant la super-approximation et l'extrapolation de l'élement fini de Han pour les équations de Stokes

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A B S T R A C T

We analyze the supercloseness properties of the nonconforming quadrilateral Han finite element for the Stokes equations. It is shown that the difference between the discrete solution and the natural interpolation of the continuous solution does not have the supercloseness property. Based on this analysis, we propose a modified interpolation operator which allows for such a result. It is then used to obtain a simple third-order extrapolation scheme.

R É S U M É

Nous présentons une analyse de la super-convergence de l'espace d'élément finis de Han pour les équations de Stokes. Il est démontré que la différence entre la solution discrète et l'interpolé naturel de la solution n'est pas de l'ordre supérieur (« supercloseness »). Basé sur notre analyse, nous proposons une modification de l'opérateur d'interpolation qui possède cette propriété. Cela permet la construction d'un schéma d'extrapolation simple qui est de l'ordre trois.

1. Introduction

Nonconforming finite element methods on quadrilateral meshes [1,4] are well-known in computational fluid mechanics, since they are locally conservation and present advantages concerning iterative solvers [5].

Supercloseness results, which state that the difference between the finite element solution and the interpolation of the continuous solution is asymptotically smaller than the interpolation error, provide the theoretical basis for defect correction and extrapolation. Extrapolation of nonconforming finite elements is a well-studied subject; see for example [2,6,3]. However, results on super-convergence of the Han element for the Stokes equations seem to be missing, and we wish to fill this gap with the present short note.

We first show, that, as opposed to the case of the Poisson equation, the difference between the discrete solution and the natural interpolation of the continuous solution does not have the supercloseness property. We then propose a modified

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interpolation operator which yields second-order accuracy in the pressure and gradients of velocities. This result allows us to construct a simple third-order accurate extrapolation scheme.

Let $\Omega \subset \mathbb{R}^2$ be bounded polygonal domain and $f \in L^2(\Omega)^2$. We denote by $\langle\cdot,\cdot\rangle$ the $L^2(\Omega)$-scalar product and by $\|\cdot\|$ the associated norm; for a subdomain $K \subset \Omega$ and a segment $S \subset \Omega$ similar notation are used. Further, $|\cdot|_k$ denotes the semi-norm in $H^k(\Omega)$, $k \geqslant 1$. Let $V = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega)$, and $U = V \times Q$. For $(v,p) \in U$ and $(w,r) \in U$ we define the continuous bilinear form $a : Q \times Q \to \mathbb{R}$ by $a((v,p),(w,r)) := \langle \nabla v, \nabla w\rangle - \langle p, \operatorname{div} w\rangle + \langle \operatorname{div} v, r\rangle$. With $l(w) := \langle f, w\rangle$, the standard weak formulation of the Stokes equations homogeneous Dirichlet boundary conditions reads: Find $(v,p) \in Q$ such that for all $(w,r) \in Q$

$$a\big((v,p),(w,r)\big) = l(w). \tag{1}$$

We consider finite element meshes h composed of rectangles K with length d_{x_i} in x_i-direction, $i = 1,2$. The set of rectangles is denoted by $\mathcal{K}_h$ and the set of edges by $\mathcal{S}_h$. With a fixed choice of unit normal, $[v_h]_S$ denotes the jump over an internal edge S; in case of a boundary edge, we set $n_S = n_\Omega$ and $[v_h]_S = v_h$.

For the pressure approximation we use the space of piece-wise constants,

$$Q_h := \big\{p_h \in L_0^2(\Omega):\ p_h|_K \in \mathcal{P}^0(K) \text{ for all } K \in \mathcal{K}_h\big\},$$

whereas the velocity approximation is sought in the nonconforming space

$$V_h := \left\{v_h \in L^2(\Omega)^2:\ v_h|_K \in \mathcal{Q}(K)^2\ \forall K \in \mathcal{K}_h,\ \int_S [v_h]_S \,\mathrm{d}s = 0\ \forall S \in \mathcal{S}_h\right\}.$$

Here $\mathcal{P}^k(K)$ $(k \in \mathbb{N})$ and $\mathcal{Q}(K)$ denote the set of polynomials of maximal degree k and the vector space engendered by $\{1, x, y, x^2, y^2\}$ on K, respectively. A local basis of V_h is defined on each cell K with edges S_i, $i = 1,\dots,4$ by means of the functionals $\phi_i(v) := \int_{S_i} v$ for $i = 1,\dots,4$ and $\phi_5(v) := \int_K v$. We define the discrete gradient and divergence operators $\nabla_h : V_h \to L^2(\Omega)^{2\times 2}$ and $\operatorname{div}_h : V_h \to L^2(\Omega)$ by $(\nabla_h v_h)|_K := \nabla(v_h|_K)$ and $(\operatorname{div}_h v_h)|_K := \operatorname{div}(v_h|_K)$, respectively. Clearly, $|v_h|_{1,h} := \|\nabla_h v_h\|$ is a norm in V_h.

Let $U_h := V_h \times Q_h$. The discrete bilinear form $a_h : Q_h \times Q_h \to \mathbb{R}$ reads

$$a_h\big((v_h,p_h),(w_h,r_h)\big) := \langle \nabla_h v_h, \nabla_h w_h\rangle - \langle p_h, \operatorname{div}_h w_h\rangle + \langle \operatorname{div}_h v_h, r_h\rangle,$$

and the discrete problem under consideration is: Find $(v_h,p_h) \in U_h$ such that for all $(w_h,r_h) \in U_h$

$$a_h\big((v_h,p_h),(w_h,r_h)\big) = l(w_h). \tag{2}$$

The definition of a_h extends to the space $U \oplus U_h$, which we equip with the norm $|||(v_h,p_h)|||_h := \sqrt{|v_h|_{1,h}^2 + \|p_h\|^2}$. It is well-known that there exists a mesh-independent constant C such that

$$\sup_{(w_h,r_h)\in U_h\setminus\{0\}} \frac{a_h((v_h,p_h),(w_h,r_h))}{|||(w_h,r_h)|||_h} \geqslant C\big|\big|\big|(v_h,p_h)\big|\big|\big|_h \quad \forall (v_h,p_h) \in U_h, \tag{3}$$

since the discrete spaces fulfill the inf-sup condition for the discrete gradient operator. It is well-known that the first-order error estimate $|||(v,p) - (v_h,p_h)||| \leqslant C d_h(|u|_2 + |p|_1)$ holds, where d_h denotes the maximal cell diameter. This error estimate uses standard results for the natural interpolation operator $I_h : U \to U_h$, $I_h = (\Pi_h, J_h)$ with $\Pi_h : V \to V_h$ and $J_h : Q \to Q_h$ defined by

$$\phi_K\big(\Pi_h(v)\big) = \phi_K(v) \quad \forall K \in \mathcal{K}_h, \qquad \phi_S\big(\Pi_h(v)\big) = \phi_S(v) \quad \forall S \in \mathcal{S}_h, \qquad \phi_K\big(J_h(p)\big) = \phi_K(p) \quad \forall K \in \mathcal{K}_h.$$

As we show in the next section, the natural interpolation operator I_h does not yield a supercloseness property. However, our analysis allows us to construct a modification which yields such a result.

2. Supercloseness analysis

First we state a bound of the difference between the discrete solution and the interpolation of the continuous solution in terms of the interpolation error.

Lemma 2.1. *Let $(v,p) \in H^3(\Omega)^2 \times H^2(\Omega)$. There is a mesh-independent constant C such that*

$$C\big|\big|\big|(v_h,p_h) - I_h(v,p)\big|\big|\big|_h \leqslant \sup_{(w_h,r_h)\in U_h\setminus\{0\}} \frac{a_h((v,p) - I_h(v,p),(w_h,r_h)) + \mathcal{C}_h}{|||(w_h,r_h)|||_h}, \tag{4}$$

where the consistency error satisfies