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Numerical Analysis

A finite element time relaxation method

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ABSTRACT

We discuss a finite element time-relaxation method for high Reynolds number flows. The method uses local projections on polynomials defined on macroelements of each pair of two elements sharing a face. We prove that this method shares the optimal stability and convergence properties of the continuous interior penalty (CIP) method. We give the formulation both for the scalar convection–diffusion equation and the time-dependent incompressible Euler equations and the associated convergence results. This note finishes with some numerical illustrations.

RÉSUMÉ

Nous présentons une méthode d'éléments finis de type relaxation de temps pour les écoulements de fluides à grand nombre de Reynolds. Cette approche utilise des projections locales sur un espace de polynômes défini sur des macor-éléments pour chaque paire d'éléments adjacents à une face intérieure du maillage. Nous démontrons la stabilité et la convergence. Nous donnons des résultats pour l'équation de convection–diffusion et les équations instationnaires d'Euler. Cette note se termine par des résultats numériques.

1. Introduction

The computation of high Reynolds number flows is one of the most challenging problems in scientific computing. Indeed the nonlinear coupling of the Navier–Stokes' equations leads to small scale features in the solution even for problems with smooth data. The standard Galerkin methods, being energy conservative, will represent all the energy carrying features on the resolved scales, leading to spurious oscillations. One way to deal with this problem is to apply a filter to the continuous equation and then model the so-called Reynolds stresses, typically by a nonlinear viscosity, such as the Smagorinsky model [11]. A different more recent approach is the time-relaxation method, where a penalty term is added for the distance from the filtered to the unfiltered solution [1].

Another advocated approach is the use of stabilized finite element methods to replace the turbulence model. These methods are designed so as to have optimal convergence for smooth solutions [9,8,2,4], and to contain perturbations in an $O(h)$ region from layers. This approach is appealing for large eddy simulation since it indicates the possibility of (i) optimal convergence for smooth solutions [4]; (ii) containment of pollution caused by high frequency content due to the nonlinear coupling [5]; (iii) optimal dissipation rates in the turbulent zone [3,10]. Similar quasi-optimal convergence proofs were obtained recently for a finite element realization of the time-relaxation method [7] in the simple case of a linear transport equation.

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Here we draw upon similarities of the symmetric stabilization and time-relaxation to propose a simplified time-relaxation method. Instead of using a global H^1 projection for the construction of the filtered approximation we use a local L^2-projection, onto a local space of smoother functions. For each face F this local space is defined on the elements sharing a face.

Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain with boundary $\partial\Omega$ and exterior normal n, $d = 2, 3$. Our model problems read

$$-\epsilon\Delta u + \beta \cdot \nabla u + u = f \in L^2(\Omega) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \quad \text{and} \tag{1}$$

$$\partial_t u + (u \cdot \nabla)u + \nabla p = f \in \left[L^2(\Omega)\right]^2, \quad \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot n = 0 \text{ on } \partial\Omega,\ u(0) = u_0. \tag{2}$$

In problem (1) we take $\epsilon \in \mathbb{R}^+$ and $\beta \in [W^{1,\infty}(\Omega)]^2$ and $f \in L^2(\Omega)$ is the source term.

2. Finite element framework

Let $\{\mathcal{T}_h\}_h$, with $\mathcal{T}_h = \{K\}$, be a quasi-uniform family of meshes with $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$. We denote the set of interior faces by $\mathcal{F}_h$. For each face $F \in \mathcal{F}_h$ we introduce the macro element $\hat{K}_F$ consisting of K and $K' \in \mathcal{T}_h$ such that $F = K \cap K'$. We introduce the standard finite element space V_h of piecewise isoparametric polynomial functions v_h, such that the reference polynomial space contains the set of polynomials of maximal total degree k, P_k. Moreover we introduce the local polynomial spaces associated to each face F

$$W_l(\hat{K}_F) := \left\{w \in P_l(\hat{K}_F)\right\}, \quad l \geqslant 0.$$

We define the scalar products $(f, g)_\Omega := \int_\Omega fg\,\mathrm{d}x$ and $\langle f, g\rangle_{\partial\Omega} := \int_{\partial\Omega} fg\,\mathrm{d}s$ for L^2-functions f, g.

2.1. Time-relaxation and equivalence with interior penalty methods

Consider the abstract problem, find $u \in V$ such that $a(u, v) = (f, v)$ for all $v \in V$ with Galerkin approximation find $u_h \in V_h$ such that $a(u_h, v_h) = (f, v_h)$ for all $v_h \in V_h$. The idea of the time-relaxation method (see for instance [7] and references therein) is then to add a term $s(u, v)$ of the form

$$s(u, v) = \left(\tau^{-1}(u - Gu), v - Gv\right)_\Omega,$$

where G is some filtering operator $G : V \mapsto W$ where W is a space with functions of higher regularity. The relaxation time τ sets the dissipation rate for the scales that are filtered out by G, typically corresponding to the eddy turnover time. The bilinear form s is symmetric and positive semi-definite.

The spurious oscillations that appears in Galerkin methods are due to energy accumulation on the highest frequencies. In the case of finite element methods the piecewise polynomial carries the approximation properties of the space, and the highest frequencies are represented by the singularities over element faces, i.e. jumps in the solution or its derivatives over element faces. It is this scale separation that naturally occurs in finite element methods that we wish to exploit here. Indeed for each interior face F we let the projection G_F be defined by $G_F u_h \in W_l(\hat{K}_F)$ such that

$$(G_F u_h, v_h)_{\hat{K}_F} = (u_h, v_h)_{\hat{K}_F}, \quad \forall v_h \in W_l(\hat{K}_F).$$

Clearly by the orthogonality of the projection this operator is symmetric, i.e.

$$(u_h - G_F u_h, v_h)_{\hat{K}_F} = (u_h - G_F u_h, v_h - G_F v_h)_{\hat{K}_F}. \tag{3}$$

The time-relaxation term that we propose will act on the singular part of the finite element solution only and is defined by

$$s_l(u_h, v_h) := \sum_{F\in\mathcal{F}_h} \tau_F^{-1}\left((u_h - G_F u_h), v_h\right)_{\hat{K}_F},$$

where l refers to the polynomial order of the projection space $W_l(\hat{K}_F)$. Let $\tau_F := h_F/\sigma_F$ where $\sigma_F > 0$ denotes a characteristic flow velocity, that may be chosen locally provided it remains bounded away from zero. It clearly has the physical dimension of time and corresponds to the time needed to cross K_F. The form (3) acts on the jump of u_h and all its derivatives and is therefore a generalization of the CIP method [6]. We now prove the equivalence with the multipenalty stabilization operator

$$j(u_h, v_h) := \sum_{F\in\mathcal{F}_h} \sum_{i=1}^{k} \tau_F^{-1} h_K^{2i+1} \int_F \left[D^i u_h\right]\left[D^i v_h\right] \mathrm{d}s,$$

where $[x]$ denotes the jump of quantity x over the face F. The sign is irrelevant.