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Complex Analysis

Some preserving sandwich results of certain integral operators on multivalent meromorphic functions

Quelques résultats de conservation de la subordination pour certains opérateurs intégraux sur les fonctions méromorphes multi-valuées

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ABSTRACT

In this paper, we obtain some subordination, superordination and sandwich-preserving results of a certain integral operator on p-valent meromorphic functions.

RÉSUMÉ

Nous présentons des résultats de sub- et super-ordination simultanées pour certains opérateurs sur les fonctions méromorphes p-valuées.

1. Introduction

Let $H(\mathbb{U})$ be the class of functions analytic in $\mathbb{U} = \{z \in \mathbb{C}\colon\ |z| < 1\}$ and $H[a,n]$ be the subclass of $H(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $H_0 = H[0,1]$ and $H = H[1,1]$. Let $\sum_p$ denote the class of all p-valent meromorphic functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad \bigl(p \in \mathbb{N} = \{1,2,3,\ldots\};\ z \in \mathbb{U}^* = \mathbb{U}\backslash\{0\}\bigr). \tag{1.1}$$

Let f and F be members of $H(\mathbb{U})$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in \mathbb{U})$, such that $f(z) = F(\omega(z))$. In such a case, we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (see [4,5]).

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first-order differential subordination:

$$\phi\bigl(p(z), zp'(z); z\bigr) \prec h(z), \tag{1.2}$$

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then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi(p(z), zp'(z); z)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first-order differential superordination:

$$h(z) \prec \phi\big(p(z), zp'(z); z\big), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [4,5]).

For a function f in the class $\sum_p$ given by (1.1), Aqlan et al. [1] introduced the following one-parameter family of integral operators:

$$\mathcal{P}_p^\alpha f(z) = \frac{1}{z^{p+1}\Gamma(\alpha)} \int_0^z \left(\log\frac{z}{t}\right)^{\alpha-1} t^{\alpha-1} f(t)\, \mathrm{d}t \quad (\alpha > 0;\ p \in \mathbb{N}). \tag{1.4}$$

Using an elementary integral calculus, it is easy to verify that (see [1]):

$$\mathcal{P}_p^\alpha f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{1}{k+p+1}\right)^\alpha a_k z^k \quad (\alpha \geqslant 0;\ p \in \mathbb{N}). \tag{1.5}$$

Also, it is easily verified from (1.5) that

$$z\big(\mathcal{P}_p^\alpha f(z)\big)' = \mathcal{P}_p^{\alpha-1} f(z) - (1+p)\mathcal{P}_p^\alpha f(z). \tag{1.6}$$

To prove our results, we need the following definitions and lemmas.

Definition 1. (See [4].) Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{\mathbb{U}} \backslash E(q)$ where

$$E(q) = \Big\{\zeta \in \partial\mathbb{U}\colon \lim_{z \to \zeta} q(z) = \infty\Big\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}$.

Definition 2. (See [5].) A function $L(z,t)$ $(z \in \mathbb{U}, t \geqslant 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqslant 0$, $L(z,0)$ is continuously differentiable on $[0;1)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leqslant t_1 \leqslant t_2$.

Lemma 1. *(See [6].) The function $L(z,t) : \mathbb{U} \times [0;1) \longrightarrow \mathbb{C}$, of the form:*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \quad \big(a_1(t) \neq 0;\ t \geqslant 0\big),$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U}, t \geqslant 0).$$

Lemma 2. *(See [2].) Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition:*

$$\mathrm{Re}\big\{H(is;t)\big\} \leqslant 0$$

for all real s and for all $t \leqslant -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\big\{H\big(p(z); zp'(z)\big)\big\} > 0 \quad (z \in \mathbb{U}),$$

then $\mathrm{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$.

Lemma 3. *(See [3].) Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(\mathbb{U})$ with $h(0) = c$. If $\mathrm{Re}\{\kappa h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad \big(z \in \mathbb{U};\ q(0) = c\big)$$

is analytic in $\mathbb{U}$ and satisfies $\mathrm{Re}\{\kappa h(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.