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Differential Geometry

Hebey–Vaugon conjecture II

La conjecture de Hebey–Vaugon II

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ABSTRACT

In this Note, we consider the remaining cases of Hebey–Vaugon conjecture. Assuming the positive mass theorem, we give a positive answer to this conjecture.

RÉSUMÉ

Dans cette Note, on considère les cas restants de la conjecture de Hebey–Vaugon. En admettant la théorème de la masse positive, on donne une réponse positive à cette conjecture.

Let (M,g) be a compact Riemannian manifold of dimension $n \geqslant 3$. Denote by $I(M,g)$, $C(M,g)$ and R_g the isometry group, the conformal transformations group and the scalar curvature, respectively. Let G be a subgroup of the isometry group $I(M,g)$. The equivariant Yamabe problem can be formulated as follows: *in the conformal class of g, there exists a G-invariant metric with constant scalar curvature*. Assuming the positive mass theorem and the Weyl vanishing conjecture (for more details on the subject, see [5,10] and the references therein), E. Hebey and M. Vaugon [4] proved that this problem has solutions. Moreover, they proved that the infimum of Yamabe functional

$$I_g(\varphi) = \left(\int_M |\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \,\mathrm{d}v \right) \|\varphi\|^{-2}_{\frac{2n}{n-2}}, \tag{1}$$

over G-invariant nonnegative functions is achieved by a smooth positive G-invariant function. This function is a solution of the Yamabe equation, which is the Euler–Lagrange equation of I_g:

$$\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi = \mu \varphi^{\frac{n+2}{n-2}}.$$

One of the consequences of these results is that the following conjecture due to Lichnerowicz [7] is true:

Lichnerowicz conjecture. *For every compact Riemannian manifold (M,g) which is not conformal to the unit sphere S^n endowed with its standard metric g_s, there exists a metric $\tilde g$ conformal to g for which $I(M,\tilde g) = C(M,g)$, and the scalar curvature $R_{\tilde g}$ is constant.*

The classical Yamabe problem, which consists of finding a conformal metric with constant scalar curvature on a compact Riemannian manifold, is a particular case of the equivariant Yamabe problem (it corresponds to $G = \{\mathrm{id}\}$). This problem

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was completely solved by H. Yamabe [13], N. Trudinger [12], T. Aubin [1] and R. Schoen [11]. The main idea to prove the existence of positive minimizers for I_g is to show that if (M,g) is not conformal to the sphere endowed with its standard metric, then

$$\mu(g) := \inf_{C^\infty(M)} I_g(\varphi) < \frac{1}{4}n(n-2)\omega_n^{2/n}, \tag{2}$$

where ω_n is the volume of the unit sphere S^n.

T. Aubin [1] proved (2) in some cases by constructing a test function u_ε satisfying $I_g(u_\varepsilon) < \frac{1}{4}n(n-2)\omega_n^{2/n}$. He conjectured that (2) always holds except for the sphere. R. Schoen constructed another test function which involves the Green function of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}R_g$. Using the positive mass theorem, R. Schoen proved (2) for all compact manifolds which are not conformal to (S^n, g_s). The solution of the Yamabe problem follows.

Later, E. Hebey and M. Vaugon [4] showed that we can generalize (2) for the equivariant case as follows:

Denote by $O_G(P)$ the orbit of $P \in M$ under G and by $\operatorname{card} O_G(P)$ its cardinality. Let $C_G^\infty(M)$ be the set of smooth G-invariant functions and

$$\mu_G(g) := \inf_{C_G^\infty(M)} I_g(\varphi).$$

Following E. Hebey and M. Vaugon [3,4], we define the integer $\omega(P)$ at a point P as

$$\omega(P) = \inf\{i \in \mathbb{N} / \|\nabla^i W_g(P)\| \neq 0\} \quad (\omega(P) = +\infty \text{ if } \forall i \in \mathbb{N},\ \|\nabla^i W_g(P)\| = 0).$$

Hebey–Vaugon conjecture. *Let (M,g) be a compact Riemannian manifold of dimension $n \geqslant 3$ and G be a subgroup of $I(M,g)$. If (M,g) is not conformal to (S^n, g_s) or if the action of G has no fixed point, then the following inequality holds*

$$\mu_G(g) < \frac{1}{4}n(n-2)\omega_n^{2/n}\Big(\inf_{Q\in M} \operatorname{card} O_G(Q)\Big)^{2/n}. \tag{3}$$

E. Hebey and M. Vaugon showed that if this conjecture holds, then it implies that the equivariant Yamabe problem has minimizing solutions and the Lichnerowicz conjecture is also true. Notice that if $G = \{\mathrm{id}\}$, then this conjecture corresponds to (2).

Assuming the positive mass theorem, E. Hebey and M. Vaugon [4] proved the following:

Theorem 1 *(E. Hebey and M. Vaugon). The Hebey–Vaugon conjecture holds if at least one of the following conditions is satisfied*:

1. *The action of G on M is free.*
2. $3 \leqslant \dim M \leqslant 11$.
3. *There exists a point $P \in M$ with finite minimal orbit under G such that $\omega(P) > (n-6)/2$ or $\omega(P) \in \{0,1,2\}$.*

The main result of this note is the following:

Theorem 2. *If there exists a point $P \in M$ such that $\omega(P) \leqslant (n-6)/2$, then*

$$\mu_G(g) < \frac{1}{4}n(n-2)\omega_n^{2/n}\big(\operatorname{card} O_G(P)\big)^{2/n}. \tag{4}$$

Note that if we assume the positive mass theorem, then Theorem 1 and Theorem 2 implies that the Hebey–Vaugon conjecture holds. In particular, it holds if M is a spin manifold.

The proof of Theorem 2 doesn't require the positive mass theorem. If $\operatorname{card} O_G(Q) = +\infty$ for all $Q \in M$, then (3) holds. So we have to consider only the case when there exists a point in M with finite orbit. From now on, we suppose that $P \in M$ is contained in a finite orbit and $\omega(P) \leqslant \frac{n-6}{2}$. The assumption $\omega(P) \leqslant \frac{n-6}{2}$ deletes the case (M,g) is conformal to (S^n, g_s).

In order to prove Theorem 2, we construct from the function $\varphi_{\varepsilon,P}$ defined below a G-invariant test function ϕ_ε such that

$$I_g(\phi_\varepsilon) < \frac{1}{4}n(n-2)\omega_n^{2/n}\big(\operatorname{card} O_G(P)\big)^{2/n}. \tag{5}$$

Let us recall the construction in [9] of $\varphi_{\varepsilon,P}$. Let $\{x^j\}$ be the geodesic normal coordinates in the neighborhood of P and define $r = |x|$ and $\xi^j = x^j/r$. Without loss of generality, we suppose that $\det g = 1 + O(r^N)$, with $N > 0$ sufficiently large (for the existence of such coordinates for a G-invariant conformal class, see [4,6]).

$$\varphi_{\varepsilon,P}(Q) = \begin{cases} (1 - r^{\omega(P)+2} f(\xi))\Big[\big(\frac{\varepsilon}{r^2+\varepsilon^2}\big)^{\frac{n-2}{2}} - \big(\frac{\varepsilon}{\delta^2+\varepsilon^2}\big)^{\frac{n-2}{2}}\Big] & \text{if } Q \in B_P(\delta); \\ 0 & \text{if } Q \in M - B_P(\delta), \end{cases}$$