C. R. Acad. Sci. Paris, Ser. I 350 (2012) 861–865

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical Problems in Mechanics

Some unilateral Korn inequalities with application to a contact problem with inclusions

Quelques inégalités de Korn unilatérales et leur application à un problème de contact avec inclusions

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ARTICLE INFO

Article history:
Received 12 July 2012
Accepted 4 September 2012
Available online 12 October 2012

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ABSTRACT

In the first part of this note, some unilateral inequalities of the Korn type are established. These inequalities seem to be new.
In the second part, these inequalities are used in an essential way to prove the existence of a solution (which is not necessarily unique) for a unilateral contact problem involving a matrix material with inclusions of various shapes (the conditions depend on the shape of each inclusion in a remarkable way).

RÉSUMÉ

Dans une première partie sont démontrées quelques inégalités de Korn unilatérales qui semblent nouvelles.
Ces inégalités sont alors utilisées de façon essentielle dans la démonstration de l'existence d'une solution pour un problème de contact unilatéral dans une matrice avec des inclusions de diverses formes (les conditions dépendent de façon remarquable de la forme de chaque inclusion).
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Version française abrégée

Dans la première partie de cette note, on démontre quelques inégalités de Korn unilatérales qui semblent nouvelles (Propositions 2.8 et 2.9), ne faisant intervenir sur le bord de l'ouvert considéré que la partie positive de la composante normale de la déformation (et, uniquement pour des inclusions de forme sphérique ou cylindrique, la composante tangentielle de la déformation).

Ces inégalités sont utilisées dans la seconde partie pour traiter un problème de contact unilatéral dans une matrice avec des inclusions de diverses formes. Il s'agit alors d'une inequation variationnelle de type Signorini sans frottement ou avec frottement de type Tresca (Problem $\mathcal{P}'$). Celle-ci s'exprime de manière équivalente sous forme de la minimisation d'une fonctionnelle convexe sur un convexe fermé (Problem $\mathcal{P}$).

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.09.006

Le résultat d'existence est donné dans la Proposition 3.2. Les hypothèses spécifiques portent sur le second membre uniquement dans les inclusions de forme sphérique ou cylindrique (hypothèses $(\mathcal{H}^j)$). L'unicité n'est pas nécessairement garantie dans les inclusions à cause de l'absence de stricte convexité de la fonctionnelle qui est minimisée.

1. Introduction

Contact problems in the framework of three-dimensional linear elasticity have been studied by many authors. The first treatment of the subject appeared in the papers of Fichera [3–5]. We also refer to the book of G. Duvaut and J.-L. Lions [2] and references therein. More recently J. Nečas and his co-workers have studied the numerical approximation of such problems [7]. See the book of I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek [8] for references on the subject.

In this note, we consider a Signorini problem with possible Tresca friction for inclusions surrounded by a matrix. The problem here differs from the previous references insofar as the contact conditions are posed on the full boundaries of each inclusion. The conditions for the existence of the solution(s) are related to those of [3–5], but explicitly involve the moments of the right-hand side with respect to spherical and cylindrical inclusions only (hypotheses $(\mathcal{H}^j)$).

The result is based on a series of unilateral Korn inequalities which seem to be new and which are adapted to the various shapes of inclusions.

1.1. Notations

- The normal component of a vector field v on the boundary of a domain is denoted v_ν, while the tangential component $v - v_\nu \nu$ is denoted v_τ (here ν is the outward unit normal to the boundary of the domain);
- the strain tensor of a vector field v is denoted by $e(v)$;
- the kernel of the strain operator e in a connected domain is the six-dimensional space of rigid motions denoted $\mathcal{R}$:

$$\mathcal{R} \doteq \{x \mapsto v_{a,b}(x) = a \wedge x + b;\ a \text{ and } b \in \mathbb{R}^3\}. \tag{1}$$

2. Korn inequalities

2.1. Classical Korn inequalities

Definition 2.1. A bounded domain O is a *Korn domain* whenever it satisfies the second Korn inequality, i.e., if there exists a constant C such that

$$\forall v \in H^1(O), \quad |v|_{H^1(O)} \leqslant C\big(|v|_{L^2(O)} + \big|e(v)\big|_{L^2(O)}\big). \tag{2}$$

It is known (see Gobert [6]) that this holds for every connected bounded domain with Lipschitz boundary, but is also true for more general domains since a finite union of Korn domains is a Korn domain.

Definition 2.2. A bounded and connected domain O is a *Korn–Wirtinger* domain whenever it satisfies the following Korn-type inequality (similar to the Poincaré–Wirtinger inequality for scalar fields):

there exists a constant C such that for every $v \in H^1(O)$ there is an element $r_v \in \mathcal{R}$ with

$$|v - r_v|_{H^1(O)} \leqslant C\big|e(v)\big|_{L^2(O)}. \tag{3}$$

Examples of Korn–Wirtinger domains are given by the following proposition which is proved by a standard contradiction argument.

Proposition 2.3. *Let O be a* connected *Korn domain for which the embedding from $H^1(O)$ into $L^2(O)$ is* compact. *Then O is a* Korn–Wirtinger *domain.*

The following is a straightforward application of inequality (3):

Corollary 2.4. *Let O be a Korn–Wirtinger domain and Γ be a closed subset of ∂O with non-zero boundary measure. Then the first Korn inequality is satisfied for vector fields which vanish on Γ, i.e. there exists a constant C such that*

$$\forall v \in H^1(O; \Gamma), \quad |v|_{H^1(O)} \leqslant C\big|e(v)\big|_{L^2(O)}. \tag{4}$$