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Complex Analysis

Oka maps

Les applications d'Oka

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ABSTRACT

We prove that for a holomorphic submersion of reduced complex spaces, the basic Oka property implies the parametric Oka property. It follows that a stratified subelliptic submersion, or a stratified fiber bundle whose fibers are Oka manifolds, enjoys the parametric Oka property.

RÉSUMÉ

Nous prouvons que, pour une submersion holomorphe des espaces complexes réduits, la propriété d'Oka simple implique la propriété d'Oka paramétrique. En particulier, toute submersion sous-elliptique stratifié possède la propriété d'Oka paramétrique.

1. Oka properties of holomorphic maps

Let E and B be reduced complex spaces. A holomorphic map $\pi : E \to B$ is said to enjoy the *Basic Oka Property* (BOP) if, given a holomorphic map $f : X \to B$ from a reduced Stein space X and a continuous map $F_0 : X \to E$ satisfying $\pi \circ F_0 = f$ (a *lifting* of f) such that F_0 is holomorphic on a closed complex subvariety X' of X and in a neighborhood of a compact $\mathcal{O}(X)$-convex subset K of X, there is a homotopy of liftings $F_t : X \to E$ $(t \in [0,1])$ of f to a holomorphic lifting F_1 such that for every $t \in [0,1]$, F_t is holomorphic in a neighborhood of K (independent of t), $\sup_{x\in K} \operatorname{dist}(F_t(x), F_0(x)) < \epsilon$, and $F_t|_{X'} = F_0|_{X'}$ (the homotopy is fixed on X').

By definition, a complex manifold Y enjoys BOP if and only if the trivial map $Y \to$ *point* does. This is equivalent to several other properties, from the simplest *Convex Approximation Property* (CAP) to the *Parametric Oka Property* (POP) concerning compact families of maps from reduced Stein spaces to Y [2]. A complex manifold enjoying these equivalent properties is called an *Oka manifold* [2,11]; these are precisely the *fibrant complex manifolds* in Lárusson's model category [9]. Here we prove that BOP $\Rightarrow$ POP also holds for holomorphic submersions. (The submersion condition corresponds to requiring smoothness as part of the definition of a variety being Oka. The singular case is rather problematic.)

Theorem 1.1. *For every holomorphic submersion $\pi : E \to B$ of reduced complex spaces, the basic Oka property implies the parametric Oka property.*

Recall [9] that a holomorphic map $\pi : E \to B$ enjoys the *Parametric Oka Property* (POP) if for any triple (X, X', K) as above and for any pair $P_0 \subset P$ of compact subsets in an Euclidean space $\mathbb{R}^m$ the following holds. Given a continuous

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map $f : P \times X \to B$ that is *X-holomorphic* (that is, $f(p, \cdot) : X \to B$ is holomorphic for every $p \in P$) and a continuous map $F_0 : P \times X \to E$ such that (a) $\pi \circ F_0 = f$, (b) $F_0(p, \cdot)$ is holomorphic on X for all $p \in P_0$ and is holomorphic on $K \cup X'$ for all $p \in P$, there exists for every $\epsilon > 0$ a homotopy of continuous liftings $F_t : P \times X \to E$ of f to an X-holomorphic lifting F_1 such that the following hold for all $t \in [0, 1]$:

(i) $F_t = F_0$ on $(P_0 \times X) \cup (P \times X')$, and
(ii) F_t is X-holomorphic on K and $\sup_{p \in P, x \in K} \mathrm{dist}(F_t(p, x), F_0(p, x)) < \epsilon$.

A stratified subelliptic holomorphic submersion, or a stratified fiber bundle with Oka fibers, enjoys BOP [3,4]. Hence Theorem 1.1 implies:

Corollary 1.2.

(i) *Every stratified subelliptic submersion enjoys POP.*
(ii) *Every stratified holomorphic fiber bundle with Oka fibers enjoys POP.*

If $\pi : E \to B$ enjoys the Oka property then by considering liftings of constant maps $X \to b \in B$ we see that every fiber $E_b = \pi^{-1}(b)$ is an Oka manifold. For stratified fiber bundles the converse holds by Corollary 1.2.

Question: *Does every holomorphic submersion with Oka fibers enjoys the Oka property?*

A holomorphic map is said to be an *Oka map* if it is a topological (Serre) fibration and it enjoys POP. Such maps are *intermediate fibrations* in Lárusson's model category [9,10]. Corollary 1.2 implies:

Corollary 1.3.

(i) *Every holomorphic fiber bundle projection with Oka fiber is an Oka map.*
(ii) *A stratified subelliptic submersion, or a stratified holomorphic fiber bundle with Oka fibers, is an Oka map if and only if it is a Serre fibration.*

Corollary 1.2(i) and the proof by Ivarsson and Kutzschebauch [8] give the following solution of the parametric Gromov–Vaserstein problem [7,12].

Theorem 1.4. *Assume that X is a finite-dimensional reduced Stein space, P is a compact subset of $\mathbb{R}^m$, and $f : P \times X \to \mathrm{SL}_n(\mathbb{C})$ is a null-homotopic X-holomorphic mapping. Then there exist a natural number N and X-holomorphic mappings $G_1, \ldots, G_N : P \times X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(p, x) = \begin{pmatrix} 1 & 0 \\ G_1(p, x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(p, x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(p, x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

2. Reduction of Theorem 1.1 to an approximation property

Assume that $\pi : E \to B$ enjoys BOP and that $(X, X', K, P, P_0, f, F_0)$ are as in the definition of POP, with $P_0 \subset P \subset \mathbb{R}^m \subset \mathbb{C}^m$. Set

$$Z = \mathbb{C}^m \times X \times E, \qquad Z_0 = \mathbb{C}^m \times X \times B, \qquad \psi = (\mathrm{id}_{\mathbb{C}^m \times X}) \times \pi : Z \to Z_0. \tag{1}$$

Observe that ψ enjoys BOP (resp. POP) if and only if π does. To the map $f : P \times X \to B$ we associate the X-holomorphic section

$$g : P \times X \to Z_0, \qquad g(p, x) = \big(p, x, f(p, x)\big) \quad (p \in P,\ x \in X), \tag{2}$$

and to the π-lifting $F_0 : P \times X \to E$ of f we associate the section

$$G_0 : P \times X \to Z, \qquad G_0(p, x) = \big(p, x, F_0(p, x)\big) \quad (p \in P,\ x \in X). \tag{3}$$

Then $\psi \circ G_0 = g$, G_0 is X-holomorphic over $K \cup X'$, and $G_0|_{P_0 \times X}$ is X-holomorphic. We must find a homotopy $G_t : P \times X \to Z$ ($t \in [0, 1]$) such that $\psi \circ G_t = g$ for all $t \in [0, 1]$, G_1 is X-holomorphic, and for all $t \in [0, 1]$ the map G_t has the same properties as G_0, G_t is uniformly close to G_0 on $K \times P$, and $G_t = G_0$ on $(P_0 \times X) \cup (P \times X')$. Set

$$Q = [0, 1] \times P, \qquad Q_0 = \big(\{0\} \times P\big) \cup \big([0, 1] \times P_0\big).$$

The following result is the key to the proof of Theorem 1.1.