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Partial Differential Equations

New characterization of the kernel of the n-dimensional Laplace operator in exterior domains

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Abstract

In this Note, we study the characterization of the kernel of the Laplace operator with Dirichlet boundary conditions in exterior domains. We consider data in weighted Sobolev spaces. ***To cite this article: C. Amrouche, Huy Hoang Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Nouvelle caractérisation du noyau du laplacien en domaine extérieur. Nous étudions dans cette Note la caractérisation du noyau de l'opérateur laplacien avec des conditions de Dirichlet au bord dans un ouvert extérieur. Nous considérons des données dans des espaces de Sobolev avec poids. ***Pour citer cet article : C. Amrouche, Huy Hoang Nguyen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let Ω' be a bounded open region of $\mathbb{R}^n$ ($n \geqslant 2$), not necessarily connected, with a Lipschitz-continuous boundary Γ and let Ω be the complement of $\overline{\Omega'}$. We suppose that Ω' has a finite number of connected components and each connected component has a connected boundary, so that Ω is connected. For convenience, the origin of the coordinate frame is attached to $\overline{\Omega'}$. The purpose of this Note is to characterize the kernel $\mathcal{A}^{p,q}(\Omega)$ of the Laplace operator with Dirichlet boundary conditions:

$$\mathcal{A}^{p,q}(\Omega) = \left\{ z \in W_0^{1,p}(\Omega) + W_0^{1,q}(\Omega);\ \Delta z = 0 \text{ in } \Omega \text{ and } z = 0 \text{ on } \Gamma \right\}. \tag{1}$$

The motivation for studying the space $\mathcal{A}^{p,q}(\Omega)$ is the regularity problem of Laplace equation. Let $f \in W_0^{-1,p}(\Omega)$, $g \in W^{1-\frac{1}{p},p}(\Gamma)$ and $u \in W_0^{1,p}(\Omega)$ be a solution of the following system:

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \Gamma.$$

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Recall that a solution u exists and is unique if and only if f and g satisfy the compatibility condition: for any $\varphi \in \mathcal{A}^{p'}(\Omega)$,

$$\langle f, \varphi\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)} = \left\langle g, \frac{\partial\varphi}{\partial \boldsymbol{n}}\right\rangle_{W^{1-\frac{1}{p},p}(\Gamma)\times W^{\frac{-1}{p},p'}(\Gamma)}. \tag{2}$$

If, in addition, $f \in W_0^{-1,q}(\Omega)$, $g \in W^{1-\frac{1}{q},q}(\Gamma)$ with $p < q$ satisfying the compatibility condition (2) by replacing p by q, the question “Does the solution u belong to $W_0^{1,q}(\Omega)$?” arises. Since there exists $v \in W_0^{1,q}(\Omega)$ satisfying $-\Delta v = f$ in Ω and $v = g$ on Γ, from Theorem 2.1 we obtain $u - v \in \mathcal{A}^{p,q}(\Omega)$. Therefore, if $q < n$ or $q = n = 2$, then $u = v$ and $u \in W_0^{1,q}(\Omega)$. Otherwise, $u = v + \lambda \in W_0^{1,q}(\Omega)$ with $\lambda \in \mathcal{A}^{p,q}(\Omega)$.

Since the problem is posed in a n-dimensional exterior domain, it is important to specify the behavior at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces which provide a correct functional setting for unbounded domains (see [2] for more details). It means that the growth and decay of functions at infinity are expressed by means of weights, in particular, the function in these weighted Sobolev spaces satisfies an optimal weighted Poincaré-type inequality. In the whole text, bold characters are used for vector or matrix fields. We now introduce the definition of weighted Sobolev spaces and some its properties. A typical point in $\mathbb{R}^n$ is denoted by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and its norm is given by $r = |\boldsymbol{x}| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. We define the weight function $\rho(\boldsymbol{x}) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. We now define the weighted Sobolev space $W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{w} \in L^p(\Omega), \nabla u \in \mathbf{L}^p(\Omega)\}$, where

$$w = \begin{cases} (1+r) & \text{if } p \neq n, \\ (1+r)\ln(2+r) & \text{if } p = n. \end{cases}$$

This space is a reflexive Banach space when endowed with the norm: $\|u\|_{W_0^{1,p}(\Omega)} = (\|\frac{u}{w}\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{\mathbf{L}^p(\Omega)})^{1/p}$. We note that the logarithmic weight only appears if $p = n$ and all the local properties of $W_0^{1,p}(\Omega)$ coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$. We set $\mathring{W}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_0^{1,p}(\Omega)}$ and we denote the dual space of $\mathring{W}_0^{1,p}(\Omega)$ by $W_0^{-1,p'}(\Omega)$, which is a space of distributions. When $\Omega = \mathbb{R}^n$, we have $W_0^{1,p}(\mathbb{R}^n) = \mathring{W}_0^{1,p}(\mathbb{R}^n)$. We have the algebraic and topological embeddings $W_0^{1,p}(\Omega) \hookrightarrow W_{-1}^{0,p}(\Omega)$ if $p \neq n$, where $W_{-1}^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \frac{u}{1+r} \in L^p(\Omega)\}$. For all $\lambda \in \mathbb{N}^n$ where $0 \leqslant |\lambda| \leqslant 2$, the mapping $u \in W_0^{1,p}(\Omega) \to \partial^\lambda u \in W_0^{1-|\lambda|,p}(\Omega)$ is continuous. Also recall the following Sobolev embeddings (see [1]): $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-p}$ and $1 < p < n$. Note that $\mathbb{R} \subset W_0^{1,p}(\Omega)$ if and only if $p \geqslant n$. We next set $\mathcal{A}^p(\Omega) = \{y \in W_0^{1,p}(\Omega);\ \Delta y = 0 \text{ in } \Omega \text{ and } y = 0 \text{ on } \Gamma\}$. In the two-dimensional space, let $U = \frac{1}{2\pi}\ln r$ be the fundamental solution of Laplace’s equation. We now define

$$u_0 = U * \left(\frac{1}{|\Gamma|}\delta_\Gamma\right), \tag{3}$$

where δ_Γ is the distribution defined by $\forall\varphi \in \mathcal{D}(\mathbb{R}^2)$, $\langle\delta_\Gamma, \varphi\rangle = \int_\Gamma \varphi\, \mathrm{d}\sigma$.

The next lemma characterizes the kernel $\mathcal{A}^p(\Omega)$ (see [3]).

Lemma 1.1. *Let* $1 < p < \infty$ *and suppose that* Γ *is of class* $C^{1,1}$.

(i) *If* $p < n$ *or if* $p = n = 2$, *then* $\mathcal{A}^p(\Omega) = \{0\}$.

(ii) *If* $p \geqslant n \geqslant 3$, *then* $\mathcal{A}^p(\Omega) = \{c(\lambda - 1);\ c \in \mathbb{R}\}$, *where* $\lambda \in \bigcap_{r > \frac{n}{n-1}} W_0^{1,r}(\Omega)$ *is the unique solution of the following problem*

$$\Delta\lambda = 0 \quad \text{in } \Omega \quad \text{and} \quad \lambda = 1 \quad \text{on } \Gamma. \tag{4}$$

(iii) *If* $p > n = 2$, *then* $\mathcal{A}^p(\Omega) = \{c(\mu - u_0);\ c \in \mathbb{R}\}$, *where* u_0 *is defined by* (3) *and* μ *is the only solution in* $\bigcap_{r>2} W_0^{1,r}(\Omega)$ *of the problem*

$$\Delta\mu = 0 \quad \text{in } \Omega \quad \text{and} \quad \mu = u_0 \quad \text{on } \Gamma. \tag{5}$$