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Systèmes dynamiques/Géométrie analytique

# Rang et courbure de Blaschke des tissus holomorphes réguliers de codimension un ☆,☆☆

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**Résumé**

A tout $d$-tissu de codimension un sur une variété holomorphe $M$ de dimension $n$, $(d > n)$, nous associons un sous-ensemble analytique $S$ de $M$, qui – génériquement – a une dimension au plus égale à $n - 1$ : on dit alors que le tissu est *régulier*.

Notant $c(n,h)$ la dimension de l'espace vectoriel des polynômes homogènes de degré $h$ à $n$ variables, nous montrons que le rang d'un tissu régulier a une borne supérieure $\pi'(n,d)$ égale à 0 pour $d < c(n,2)$, et à $\sum_{h=1}^{k_0}(d - c(n,h))$ pour $d \geqslant c(n,2)$, $k_0$ désignant l'entier tel que $c(n,k_0) \leqslant d < c(n,k_0+1)$. Cette borne est optimale pour les tissus réguliers. Elle est strictement inférieure à la borne $\pi(n,d)$ de Chern–Castelnuovo pour $n \geqslant 3$.

En outre, si $d$ est précisément égal à $c(n,k_0)$, nous définissons une connexion holomorphe sur un certain fibré vectoriel holomorphe $\mathcal{E}$ de rang $\pi'(n,d)$ au dessus de $M \setminus S$, tel que l'espace vectoriel des germes de relation abélienne du tissu en un point de $M \setminus S$ soit isomorphe à l'espace vectoriel des germes, en ce point, de sections holomorphes de $\mathcal{E}$ ayant une dérivée covariante nulle : la courbure de cette connexion, qui généralise la courbure de Blaschke–Dubourdieu–Pantazi–Hénaut, est alors l'obstruction à ce que le rang du tissu atteigne la valeur $\pi'(n,d)$. [Pour $n = 2$, $S$ est toujours vide de sorte que tout tissu est régulier, $\pi'(2,d)$ est égal à $\pi(2,d)$, et tout $d$ peut s'écrire sous la forme $c(2,k_0)$ : nous retrouvons les résultats de Pantazi (1938) et de Hénaut (2004).]

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**Abstract**

**Rank and curvature of Blaschke for regular holomorphic webs of codimension one.**[1] To any $d$-web of codimension one on a holomorphic $n$-dimensional manifold $M$ $(d > n)$, we associate an analytic subset $S$ of $M$. We call *regular* the webs for which $S$ has at most dimension $n - 1$. This condition is generically satisfied.

Denoting by $c(n,h)$ the dimension of the vector space of homogeneous polynomials of degree $h$ in $n$ variables, we prove that the rank of a regular web has an upper-bound $\pi'(n,d)$ equal to 0 for $d < c(n,2)$, and to $\sum_{h=1}^{k_0}(d - c(n,h))$ for $d \geqslant c(n,2)$, $k_0$ denoting the integer such that $c(n,k_0) \leqslant d < c(n,k_0+1)$. This bound $\pi'(n,d)$ is optimal for regular webs. For $n \geqslant 3$, it is strictly smaller than the bound $\pi(n,d)$ of Chern–Castelnuovo.

☆ Cet article est un résumé de [V. Cavalier, D. Lehmann, Regular holomorphic webs of codimension one, arXiv: math/0703596v1 [math. DS], 20/03/2007, [1]], sans démonstration.

☆☆ Nous avons récemment modifié la terminologie, et appelons désormais “ordinaires” les tissus dits “réguliers” dans cette note.

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[1] We changed recently the terminology, and call now “ordinary” instead of “regular” the webs which are studied in this note.

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Moreover, if $d$ is precisely equal to $c(n, k_0)$, we define a holomorphic connection on some vector bundle $\mathcal{E}$ of rank $\pi'(n, d)$ above $M \setminus S$, such that the vector space of germs of Abelian relation of the web at a point of $M \setminus S$ is isomorphic to the vector space of germs at that point of holomorphic sections of $\mathcal{E}$ with vanishing covariant derivative: the curvature of this connection, which generalizes the curvature of Blaschke–Dubourdieu–Panzani–Hénaut, is then the obstruction for the rank of the web to reach the value $\pi'(n, d)$. [When $n = 2$, $S$ is always empty so that any web is regular, $\pi'(2, d) = \pi(2, d)$, any $d$ may be written $c(2, k_0)$: we recover the results given in Pantazi (1938) and Hénaut (2004).]

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## Abridged English version

A holomorphic $d$-web $W$ of codimension one on a $n$-dimensional holomorphic manifold $M$ being given ($d > n$), we denote by $M_0$ the open set in $M$ where the web is locally defined by $d$ holomorphic foliations $\mathcal{F}_i$ of codimension one, all non-singular and with tangent spaces to the leaves distinct at any point. We shall say that the web is in *strong general position* if *any* subset of $n$ leaves among the $d$ leaves through a point of $M_0$ are in general position. If we assume only that *there exists* a subset of $n$ leaves among the $d$ leaves through a point of $M_0$ which are in general position (but not necessarily any $n$ of them), we shall say that the web is in *weak general position*.

The rank of the web at a point $m$ of $M_0$ is the dimension of the vector space of germs of Abelian relations at that point. If the web is in strong general position, Hénaut proved [5] that the rank at a point does not depend on this point. When we require only the web to be in weak general position we shall call "rank of the web" the maximum of the rank at each point of $M_0$. [Notice that, in this case, the rank at a point is an upper-semicontinuous function of the point.]

When the web is in strong general position, its rank is always upper-bounded, after Chern [2], by the number $\pi(n, d)$ of Castelnuovo (the maximum of the arithmetical genus of an irreducible non-degenerate algebraic curve of degree $d$ in the $n$-dimensional complex projective space $\mathbb{P}_n$). Moreover, the rank of an algebraic $d$-web in $\mathbb{P}_n$ (i.e. the web whose leaves are the hyperplanes belonging to some algebraic curve $\Gamma$ of degree $d$ in the dual projective space $\mathbb{P}'_n$) is equal to the arithmetical genus of $\Gamma$: this is, after duality, a theorem coming back to Abel [3]. Therefore, the bound $\pi(n, d)$ is optimal for webs in strong general position.

When $n = 2$, the obstruction for the web to have maximal rank $\pi(2, d) = (d - 1)(d - 2)/2$ is the Blaschke–Hénaut curvature, which has been defined by Blaschke–Dubourdieu [10] for $d = 3$, and generalized independently by Pantazi [6] and Hénaut [4] for any $d \geqslant 3$.

In this Note, we define, for any $d$-web which is in weak general position, an analytical subset $S$ of $M_0$ which has generically dimension at most $n - 1$: in this case, the web will be said "*regular*".

Let $c(n, h)$ be the dimension of the vector space of homogeneous polynomials of degree $h$ in $n$ variables, and denote by $k_0$ the integer ($\geqslant 1$) such that $c(n, k_0) \leqslant d < c(n, k_0 + 1)$. Set

$$\pi'(n, d) = \begin{cases} 0 & \text{when } d < c(n, 2),\ (k_0 = 1), \\ \sum_{h=1}^{k_0} (d - c(n, h)) & \text{when } d \geqslant c(n, 2),\ (k_0 \geqslant 2). \end{cases}$$

The main results of this Note are then the two following:

**Theorem 1.** *The rank of any regular $d$-web on some $n$-dimensional manifold $M_0$ is at most equal to $\pi'(n, d)$. This bound is optimal.*

**Theorem 2.** *If $d = c(n, k_0)$, and if the $d$-web is regular, there exists a holomorphic vector bundle $\mathcal{E}$ of rank $\pi'(n, d)$ over $M_0 \setminus S$ and a holomorphic connection $\nabla$ on $\mathcal{E}$, such that the vector space of germs of Abelian relations at a point of $M_0 \setminus S$ is isomorphic to the vector space of germs of sections of $\mathcal{E}$ which have a vanishing covariant derivative: the curvature of this connection is then the obstruction for the rank of the web to reach the value $\pi'(n, d)$.*

For $n = 2$, it happens:

- that $S$ is always empty, so that all webs are regular,
- that the upper-bounds $\pi(2, d)$ and $\pi'(2, d)$ coincide,
- that any $d$, $d \geqslant 3$, may be written $d = c(2, k_0)$, with $k_0 = d - 1$.
- Thus, we recover the results given locally in [6,4].