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Nonsingular Ricci flow on a noncompact manifold in dimension three

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Abstract

We consider the Ricci flow $\frac{\partial}{\partial t}g = -2Ric$ on the 3-dimensional complete noncompact manifold $(M, g(0))$ with nonnegative curvature operator, i.e., $Rm \geqslant 0$, and $|Rm(p)| \to 0$, as $d(o, p) \to \infty$. We prove that the Ricci flow on such a manifold is nonsingular in any finite time. ***To cite this article: L. Ma, A. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Flot de Ricci non singulier sur une variété tridimensionnelle non compacte. Nous considérons le flot de Ricci $\frac{\partial}{\partial t}g = -2Ric$ sur la variété tridimensionnelle complète de courbure non négatif, c'est-à-dire $Rm \geqslant 0$ et $|Rm(p)| \to 0$ si $d(o, p) \to \infty$. Nous démontrons que le flot de Ricci sur une telle variété est non singular pour tout temps fini. ***Pour citer cet article : L. Ma, A. Zhu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note nous considérons le flot de la courbure de Ricci $\frac{\partial g}{\partial t}(t) = -2\,\mathrm{Ricci}_{g(t)}$ sur des variétés complètes de dimension 3 et d'opérateur de courbure positif ou nul. Nous supposons que le tenseur de Riemann, $|\mathrm{Riem}(p)|$, tend vers 0 lorsque le point p tend vers l'infini. Nous démontrons alors que le flot de Ricci est défini pour tout temps $t > 0$ et est non singulier. Ce type de questions a été posé par R. Hamilton. Sachant que l'existence en temps petit des solutions est prouvée par W.-X. Shi, il ne reste qu'à montrer que la courbure est bornée sur tout intervalle de temps fini. Plus précisément dans ce travail nous prouvons le théorème suivant :

Théorème 0.1. *Supposons que $(M, g(t))$ est un flot de Ricci pour $t \in [0, T)$ sur une variété de dimension 3 complète non compacte, connexe. Nous supposons que l'opérateur de courbure de $(M, g(0))$ est positif ou nul et vérifie $|\mathrm{Riem}(p, g(0))| \to 0$ lorsque $d(0, p) \to +\infty$. Alors $T = +\infty$, c'est-à-dire que le flot est non singulier sur tout intervalle de temps fini.*

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Par le théorème de l'âme de Cheeger–Gromoll et Meyer la variété est difféomorphe à $\mathbf{R}^3$. Rappelons deux remarques importantes :

1) Si $\mathrm{Riem}(x,t) := \mathrm{Riem}_{g(t)}(x)$ a une valeur propre nulle, alors, par le principe du maximum fort, la métrique le long du flot relevé au revêtement universel se décompose en un produit et la condition sur la courbure ne peut être satisfaite que si la métrique est constamment plate. Dans ce cas, il est évident que le flot existe pour tout temps.
2) La convergence lorsque $t \to +\infty$ est illustrée par les exemples donnés en introduction et est plus subtile que pour le cas ou t tend vers une limite finie.

Nous utilisons de manière importante les notions et idées introduites par G. Perelman dans ses travaux et nous nous appuyons sur les détails fournis dans l'ouvrage récent de J. Morgan et G. Tian.

1. Introduction

The aim of this Note is to get a global existence of Ricci flow with bounded nonnegative curvature operator in three dimensions. This kind of question was asked by Hamilton [6]. We remark that the local existence of the flow was obtained by Shi [15]. So we only need to show that the curvature is bounded in finite time. Our research is based on previous important results obtained by Hamilton and Perelman [6,9,10], which will be recalled in next section.

The Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ on a compact manifold was first introduced by Richard Hamilton [7]. Using it, Hamilton had obtained an remarkable theorem [7] that a compact 3-manifold with positive Ricci curvature can be deformed by the Ricci flow to a space form. Then we met a useful program, the so called Hamilton's program, which is to prove Poincaré conjecture and Thurston's geometrization conjecture by Ricci flow. In three remarkable papers [12–14], Perelman significantly advanced the theory of the Ricci flow. Perelman introduced important results such as a noncollapsing, canonical neighborhood, and analysis of the high curvature regions. Perelman also analyzed one of the special solution to the Ricci flow, the κ solution, which is usually the limit solution of the blow up sequence. Before the works of Perelman, Hamilton [6] had defined asymptotic volume for a complete noncompact manifold, and he had obtained that the asymptotic volume is constant under Ricci flow with bounded curvature. By an induction argument, Perelman obtained that the asymptotic volume is zero when the solution is an κ solution. In order to analyze the high curvature region, Hamilton obtained a very interesting compactness result of Ricci flow [8]. However, in order to apply this compactness, one has to check the assumptions of noncollapsing and bounded curvature. We shall use ideas above to study nonsingular Ricci flow on a complete noncompact Riemannian manifold of dimension three. The purpose of this work is to show that the following result is true:

Theorem 1.1. *Assume that $(M, g(t)), t \in [0, T)$ is a Ricci flow on the 3-dimensional connected complete noncompact Riemannian manifold $(M, g(0))$. Suppose the curvature operator of the initial metric $g(0)$ is positive, i.e., $Rm(g(0)) \geqslant 0$ with $|Rm(p, g(0))| \to 0$, $d(o, p) \to \infty$. Then $T = \infty$, i.e., Ricci flow is nonsingular in finite time on such a manifold.*

By the Soul theorem (Cheeger–Gromoll–Meyer, see Theorem 2.7 in p. 56 in [10]), each $(M, g(t))$ is diffeomorphic to $\mathbf{R}^3$.

We make two remarks here:

(1) If $Rm(x,t) := Rm(x, g(t))$ has a zero eigenvalue, then, by the strong maximum principle, we can split the flow on the level of its covering space. Then the condition $|Rm(p)| \to 0$, as $d(o, p) \to \infty$, cannot be satisfied, unless the manifold is flat; in this case the Ricci flow exists for all time (see Corollary 4.20 in [10]).
(2) We point out that in our proof of Theorem 1.1, the convergence means the geometric convergence (Definition 5.12 in p. 114 in [10]). However, as for the convergence of the global flow as $t \to \infty$, we have following interesting example, which shows that the convergence question is subtle:

Example 1.2. Consider the revolution paraboloid $x_4 = x_1^2 + x_2^2 + x_3^2$, where $(x_1, \dots, x_4) \in R^4$. We know that its curvature operator satisfies $Rm(x) \to 0$, $x \to \infty$ and $Rm > 0$. By Theorem 1.1, the Ricci flow on it can a nonsingular