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Functional Analysis

Smallest singular value of random matrices with independent columns

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Abstract

We study the smallest singular value of a square random matrix with i.i.d. columns drawn from an isotropic symmetric log-concave distribution. We prove a deviation inequality in terms of the isotropic constant of the distribution. ***To cite this article: R. Adamczak et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la plus petite valeur singulière de matrices aléatoires avec des colonnes indépendantes. On étudie la plus petite valeur singulière d'une matrice carrée aléatoire dont les colonnes sont des vecteurs aléatoires i.i.d. suivant une loi à densité log-concave isotrope. On démontre une inégalité de déviation en fonction de la constante d'isotropie. ***Pour citer cet article : R. Adamczak et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

The behaviour of the smallest singular value of random matrices with i.i.d. random entries attracted a lot of attention over the years. Major results were recently obtained in [5,8–10]. In asymptotic geometry one is interested in sampling vectors uniformly distributed in a convex body. In particular the entries are not necessarily independent. In this note, we study the more general case when the columns are i.i.d. random vectors with a symmetric isotropic log-concave

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distribution. We prove a deviation inequality for the smallest singular value in terms of a parameter L_μ which, in the case of sampling from a convex body, corresponds to the isotropic constant of the body.

Recall that a non-negative function f on $\mathbb{R}^n$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and all $\theta \in (0,1)$, $f((1-\theta)x+\theta y) \geqslant f(x)^{1-\theta} f(y)^\theta$. In this paper a symmetric probability measure μ on $\mathbb{R}^n$ is said to be log-concave if its density f is symmetric log-concave and it is called isotropic if its covariance matrix is the identity. We will also set $L_\mu = f(0)^{1/n}$. Let us observe that if μ is an isotropic probability measure uniformly distributed on a symmetric convex body K then L_μ is the so-called isotropic constant of K. If X is a random vector, distributed according to μ, we will also write $L_X = L_\mu$.

We shall use the notation $|\cdot|$ to denote the Euclidean norm of a vector or the volume or the cardinality of a set.

Theorem 1. *Let $n \geqslant 1$ and let Γ be an $n \times n$ matrix with independent columns drawn from an isotropic symmetric log-concave probability μ. For every $\varepsilon \in (0,1)$ and all $\delta \in (0,1)$ and all $M \geqslant 1$ we have*

$$\mathbb{P}\left(\inf_{x \in S^{n-1}} |\Gamma x| \leqslant \varepsilon \left(\frac{c_1}{M L_\mu}\right)^{1/(1-\delta)} n^{-1/2}\right) \leqslant \frac{C\varepsilon}{\delta} + \mathrm{e}^{-c_2 n} + \mathbb{P}\big(\|\Gamma\| > M\sqrt{n}\big), \tag{1}$$

where $c_1, c_2 > 0$ and C are absolute constants. Moreover, if $\delta \leqslant 1 - 1/(2n)$, then

$$\mathbb{P}\left(\inf_{x \in S^{n-1}} |\Gamma x| \leqslant \varepsilon \left(\frac{c_1}{M L_\mu}\right)^{1/(1-\delta)} n^{-1/2}\right) \leqslant \frac{C\varepsilon^{1/2}}{\delta} + \mathbb{P}\big(\|\Gamma\| > M\sqrt{n}\big). \tag{2}$$

Estimates for $\mathbb{P}(\|\Gamma\| > M\sqrt{n})$, when M is a power of $\log n$, can be deduced from [6] and [3].

An important case when we have more information (that follows from a result of Aubrun [1]) is that of 1-unconditional measures. Recall that a probability measure with density f is 1-unconditional if for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$, $f(x_1, \ldots, x_n) = f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Corollary 2. *If a probability μ is 1-unconditional, then Γ satisfies*

$$\mathbb{P}\left(\inf_{x \in S^{n-1}} |\Gamma x| \leqslant \varepsilon n^{-1/2}\right) \leqslant C\varepsilon + 2\mathrm{e}^{-cn^{1/5}},$$

where C and $c > 0$ are absolute constants. Moreover, for all $\varepsilon \in (0,1)$ we have

$$\mathbb{P}\left(\inf_{x \in S^{n-1}} |\Gamma x| \leqslant \varepsilon n^{-1/2}\right) \leqslant C\varepsilon^{cn^{1/5}/(2(cn^{1/5}+1))}.$$

The proof of the theorem requires the study of the isotropic constant of a sum of i.i.d. random vectors in $\mathbb{R}^n$. Let $X_1, \ldots, X_n$ be independent isotropic log-concave symmetric random vectors in $\mathbb{R}^n$. Let $x \in S^{n-1}$, and set

$$Z = x_1 X_1 + \cdots + x_n X_n.$$

Then it is well known that Z is also an isotropic log-concave symmetric random vector in $\mathbb{R}^n$. If $X_1, \ldots, X_n$ are 1-unconditional, then so is Z. The following theorem is of independent interest.

Theorem 3. *Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^n$, distributed according to a symmetric isotropic log-concave probability μ, let $x \in S^{n-1}$ and $Z = x_1 X_1 + \cdots + x_n X_n$. Then $L_Z \leqslant C L_\mu$, where C is a universal constant.*

The proof is based on the following version of a result by Gluskin and Milman [2]. Recall that K is called a star body whenever $tK \subset K$ for all $0 \leqslant t \leqslant 1$, and in such a case $\|\cdot\|_K$ denotes its Minkowski functional.

Lemma 4. *Let $f_1, \ldots, f_m$ be densities of probability measures on $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a star body containing the origin in its interior. Then for all $\lambda_1, \ldots, \lambda_m$ we have*

$$\left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \left\|\sum_{i=1}^m \lambda_i x_i\right\|_K^2 \prod_{i=1}^m f(x_i)\,\mathrm{d}x_i\right)^{1/2} \geqslant c|K|^{-1/n} \left(\sum_{i=1}^m \lambda_i^2 r_i^2\right)^{1/2}, \tag{3}$$

where $r_i^2 = \int_0^\infty |\{x\colon\ f_i(x) \geqslant t\}|^{1+2/n}\,\mathrm{d}t \geqslant \|f_i\|_\infty^{-2/n}$ and $c > 0$ is an absolute constant.