C. R. Acad. Sci. Paris, Ser. I 348 (2010) 873–877

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

Liouville-type theorems for certain degenerate and singular parabolic equations

Théorèmes de type Liouville pour quelques équations paraboliques singulières dégénérées

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ARTICLE INFO

Article history:
Received 5 April 2010
Accepted 17 June 2010
Available online 23 July 2010

Presented by Pierre-Louis Lions

ABSTRACT

Relying on recent results on Harnack inequalities for equations of p-Laplacian type, we prove Liouville-type estimates for solutions to these equations, both in the degenerate ($p > 2$), and in the singular ($1 < p < 2$) range.

RÉSUMÉ

En utilisant des résultats récents sur l'inégalité de Harnack pour les équations type p-laplacien, on établit des théorèmes de type Liouville pour les solutions de ces équations, dans le cas dégénéré $p > 2$, ainsi bien que dans le cas singulier $1 < p < 2$.

Version française abrégée

On sait que, pour les solutions de l'équation de la chaleur, des limitations unilatérales ne sont pas suffisantes pour garantir qu'elles sont constantes.

Des résultats analogues sont valables pour les solutions faibles des équations (1), (2) dans le cas $p > 2$.

Le résultat fondamental de cette Note montre que, dans l'intervalle singulier sur-critique (3), les solutions faibles de (1), (2), définies dans tout $\mathbb{R}^{N+1}$ et bornées inférieurment (ou supérieurment) sont en fait constantes (Théorème 1.2).

Ce théorème n'est plus vrai quand p est dans l'intervalle singulier critique et sous-critique (4), comme on peut voir, grâce à certaines solutions explicites de (1)′ dans cet intervalle.

Dans le cas dégénéré ($p > 2$) il est nécessaire de supposer des limitations soit inférieures soit supérieures, pour pouvoir garantir que la solution est constante. Il est possible de formuler ces limitations bilatérales de différentes façons, comme on le montre dans le Théorème 1.1 et dans les Propositions 1.1, 1.2.

1. Liouville-type theorems

For $T \in \mathbb{R}$ let S_T denote the semi-infinite strip

$$S_T = \mathbb{R}^N \times (-\infty, T).$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.06.019

Let u be a non-negative, local, weak solution to the quasi-linear parabolic equation

$$\begin{aligned} &u \in C_{\mathrm{loc}}\big(-\infty, T; L^2_{\mathrm{loc}}(\mathbb{R}^N)\big) \cap L^p_{\mathrm{loc}}\big(-\infty, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)\big), \\ &u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } S_T, \end{aligned} \tag{1}$$

for $p > 1$, where $\mathbf{A}: S_T \to \mathbb{R}^N$, is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \geqslant C_o|Du|^p \\ \big|\mathbf{A}(x,t,u,Du)\big| \leqslant C_1|Du|^{p-1} \end{cases} \quad \text{a.e. in } S_T \tag{2}$$

where C_o and C_1 are given positive constants. The prototype is

$$u_t - \operatorname{div}|Du|^{p-2}Du = 0, \quad \text{in } S_T. \tag{1$'$}$$

The modulus of ellipticity of this class of equation is $|Du|^{p-2}$ and accordingly they are degenerate for $p > 2$ and singular for $1 < p < 2$.

Harmonic functions in $\mathbb{R}^N$ with one-sided bound, are constant. This, known as the Liouville theorem, is solely a consequence of the Harnack inequality. As such it extends to solutions to homogeneous, quasi-linear, elliptic partial differential equations in $\mathbb{R}^N$ with one-sided bound.

This property does not extend to caloric functions in $\mathbb{R}^N \times \mathbb{R}$, as a one-sided bound is not sufficient to imply that they are constant. The function

$$\mathbb{R}\times\mathbb{R} \ni (x,t) \to u(x,t) = e^{x+t}$$

is a non-negative, non-constant solution of the heat equation in $\mathbb{R}\times\mathbb{R}$. The Liouville theorem continues to be false for non-negative solutions to degenerate p-Laplacian type equations ($p > 2$). The one-parameter family of non-negative functions defined in the whole $\mathbb{R}\times\mathbb{R}$

$$u(x,t;c) = A(1-x+ct)_+^{\frac{p-1}{p-2}}, \quad \text{where } A = c^{\frac{1}{p-2}}\left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

is a non-negative, non-constant, weak solution to (1)$'$ in $\mathbb{R}^2$.

The main result of this note is that the Liouville property while false for p in the degenerate range $p > 2$, it does actually holds true for p in the singular, super-critical range

$$\frac{2N}{N+1} < p < 2 \tag{3}$$

and then it is false again for p in the singular, critical, and sub-critical range

$$1 < p \leqslant \frac{2N}{N+1}. \tag{4}$$

While some results appear in the literature for linear and *coercive* equations ($p = 2$) (see for example [4–6]), to our knowledge, no results are known for degenerate ($p > 2$) or singular ($1 < p < 2$) quasi-linear equations of the type of (1)–(2).

1.1. Two-sided bounds and Liouville-type theorems in the degenerate range $p > 2$

Henceforth we let u be a continuous, local, weak solution to (1)–(2) in S_T for $p > 2$.

Theorem 1.1. *If u is bounded in S_T, then u is constant.*

The next proposition asserts that if a one-sided bound is available, then it suffices to verify the two-sided bound only at some time level.

Proposition 1.1. *Let u be bounded below in S_T and assume that*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < +\infty \quad \textit{for some } s < T.$$

Then u is constant in S_s.

It has been observed that a one-sided bound on u is not sufficient to infer that u is constant in S_T. Such a conclusion however holds if u has a two-sided bound as indicated by Theorem 1.1. Consider the family of functions

$$u(x,t) = C(N,p)\left(\frac{|x|^p}{T-t}\right)^{\frac{1}{p-2}}$$