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Géométrie algébrique

Nombres de Betti des fibres de Springer de type A

Lucas Fresse

Department of Mathematics, the Weizmann Institute of Science, 76100 Rehovot, Israel

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Présenté par Michel Duflo

Résumé

Soit u un endomorphisme nilpotent d'un espace vectoriel de dimension finie. La fibre de Springer au-dessus de u, notée $\mathcal{B}_u$, est la variété des drapeaux complets stables par u. On détermine les nombres de Betti de $\mathcal{B}_u$. Dans ce but, on construit une décomposition cellulaire de $\mathcal{B}_u$. La codimension des cellules est similaire à une longueur de Coxeter, donc notre décomposition cellulaire est adaptée au calcul des nombres de Betti. ***Pour citer cet article : L. Fresse, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Abstract

Betti numbers of Springer fibers in type *A*. Let u be a nilpotent endomorphism of a finite dimensional vector space. The Springer fiber over u, denoted by $\mathcal{B}_u$, is the variety of complete flags stable by u. We determine the Betti numbers of $\mathcal{B}_u$. To do this, we construct a cell decomposition of $\mathcal{B}_u$. The codimension of the cells is similar to a Coxeter length, this makes our cell decomposition well suited for the calculation of Betti numbers. ***To cite this article: L. Fresse, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Abridged English version

Let V be a n-dimensional $\mathbb{C}$-vector space and let $u : V \to V$ be a nilpotent endomorphism. Let $\mathcal{B}$ be the variety of complete flags on V and let $\mathcal{B}_u$ be the subset of u-stable flags, i.e. flags $(V_0, \ldots, V_n) \in \mathcal{B}$ with $u(V_i) \subset V_i$ for any i. The variety $\mathcal{B}$ is projective and $\mathcal{B}_u$ is a closed subvariety of $\mathcal{B}$. The variety $\mathcal{B}_u$ is called a *Springer fiber* since it can be seen as the fiber over u of the Springer resolution of singularity of the nilpotent cone of $\mathrm{End}(V)$ (see [2]). Our purpose is to compute the Betti numbers $\dim_{\mathbb{Q}} H^m(\mathcal{B}_u, \mathbb{Q})$.

To do this, we construct a cell decomposition of $\mathcal{B}_u$. A finite partition of an algebraic variety X is said to be an α-partition if the subsets in the partition can be indexed $X_1, \ldots, X_k$ so that $X_1 \cup \cdots \cup X_l$ is closed for any $l \leqslant k$. An α-partition is a *cell decomposition* if each subset in the partition is isomorphic as variety to an affine space. If X is projective and has a cell decomposition, then the cohomology of X vanishes in odd degrees and $\dim_{\mathbb{Q}} H^{2m}(X, \mathbb{Q})$ is equal to the number of m-dimensional cells in the decomposition. If $u = 0$ the variety $\mathcal{B}_u = \mathcal{B}$ has a cell decomposition into Schubert cells $S(\sigma)$ parameterized by the elements of the symmetric group $\sigma \in S_n$, and the codimension of $S(\sigma)$

Adresse e-mail : lucas.fresse@weizmann.ac.il.

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is the inversion number of σ. For u general, we construct a cell decomposition of $\mathcal{B}_u$ parameterized by a set of row-standard tableaux, such that the codimension of cells can also be interpreted as an inversion number.

Let $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ be the lengths of the Jordan blocks of u and let $Y(u)$ be the Young diagram of rows of lengths $\lambda_1, \ldots, \lambda_r$. If $\mu_1, \ldots, \mu_s$ are the lengths of the columns of $Y(u)$, recall from [1, §II.5.5] that $\dim_{\mathbb{C}} \mathcal{B}_u = \sum_{q=1}^{s} \mu_q(\mu_q - 1)/2$.

A standard tableau of shape $Y(u)$ is a numbering of the boxes of $Y(u)$ by $1, \ldots, n$ such that numbers in the rows increase to the right and numbers in the columns increase to the bottom.

We call *row-standard tableau of shape* $Y(u)$ a numbering by $1, \ldots, n$ of the boxes of $Y(u)$ such that numbers in the rows increase to the right. Let τ be a row-standard tableau. If we put in increasing order the entries in each column of τ, then we get a new tableau which is standard. Let us write it $\mathrm{st}(\tau)$.

We call *inversion* a pair (i, j) of numbers $i < j$ in the same column of τ and such that one of the following conditions is satisfied:

- i or j has no box on its right and i is below j,
- i, j have respective entries i', j' on their right, and $i' > j'$.

For example $\tau = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 3 & 6 & 7 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}$ has four inversions: the pairs $(1, 2)$, $(4, 6)$, $(5, 6)$, $(7, 8)$.

Let $n_{\mathrm{inv}}(\tau)$ be the number of inversions of τ. We see that $n_{\mathrm{inv}}(\tau) = 0$ if and only if τ is a standard tableau. For $u = 0$ the diagram $Y(u)$ has only one column and then τ is equivalent to a permutation ($\sigma \in S_n$ corresponds to the tableau numbered by $\sigma_1, \ldots, \sigma_n$ from top to bottom) and $n_{\mathrm{inv}}(\tau)$ is the usual inversion number for permutations.

Our main result is the following:

Theorem. *Let $d = \dim_{\mathbb{C}} \mathcal{B}_u$. The variety $\mathcal{B}_u$ has a cell decomposition $\mathcal{B}_u = \bigcup_\tau C(\tau)$ parameterized by the row-standard tableaux of shape $Y(u)$, such that* $\dim_{\mathbb{C}} C(\tau) = d - n_{\mathrm{inv}}(\tau)$.

Thus we deduce:

Corollary. *For any $m \geqslant 0$, we have $H^{2m+1}(\mathcal{B}_u, \mathbb{Q}) = 0$, and* $\dim_{\mathbb{Q}} H^{2m}(\mathcal{B}_u, \mathbb{Q})$ *is the number of row-standard tableaux τ of shape $Y(u)$ such that $n_{\mathrm{inv}}(\tau) = d - m$.*

This allows us to obtain an explicit formula for Betti numbers (see the relation (2) of Section 3.1 below).

Our construction of cells relies on a construction by Spaltenstein [1] of an α-partition of $\mathcal{B}_u$ into subsets parameterized by standard tableaux. Let T be standard. For $i = 0, \ldots, n$ the shape of the subtableau $T[1, \ldots, i]$ of entries $1, \ldots, i$ is a Young diagram Y_i^T with i boxes. Let $(V_0, \ldots, V_n) \in \mathcal{B}_u$ be a u-stable flag. For $i = 0, \ldots, n$ the restriction $u_{|V_i} : V_i \to V_i$ is nilpotent and its Jordan form is represented by a Young diagram $Y(u_{|V_i})$ with i boxes. Define

$$\mathcal{B}_u^T = \big\{(V_0, \ldots, V_n) \in \mathcal{B}_u \colon\ Y(u_{|V_i}) = Y_i^T\ \forall i = 0, \ldots, n\big\}.$$

By [1, §II.5.4–5] the subsets $\mathcal{B}_u^T$ form an α-partition of $\mathcal{B}_u$, and each $\mathcal{B}_u^T$ is irreducible and such that $\dim_{\mathbb{C}} \mathcal{B}_u^T = \dim_{\mathbb{C}} \mathcal{B}_u$. (In particular $\mathcal{B}_u$ is equidimensional and the irreducible components of $\mathcal{B}_u$ are the subsets $K^T = \overline{\mathcal{B}_u^T}$.)

For each T, we construct a cell decomposition $\mathcal{B}_u^T = \bigcup_\tau C(\tau)$ parameterized by row-standard tableaux with $\mathrm{st}(\tau) = T$, and such that the codimension of $C(\tau)$ in $\mathcal{B}_u^T$ is equal to $n_{\mathrm{inv}}(\tau)$.

Finally, by collecting together the cell decompositions of the $\mathcal{B}_u^T$'s for T running over the set of standard tableaux of shape $Y(u)$, we get a cell decomposition $\mathcal{B}_u = \bigcup_\tau C(\tau)$ satisfying the properties of the theorem.

1. Introduction

Soit V un espace vectoriel complexe de dimension $n \geqslant 0$ et soit $u : V \to V$ un endomorphisme nilpotent. On note par $\mathcal{B}$ la variété algébrique projective formée par les drapeaux complets de V. L'ensemble des drapeaux $(V_0, \ldots, V_n) \in \mathcal{B}$ tels que $u(V_i) \subset V_i$ pour tout i, que l'on note $\mathcal{B}_u$, forme une sous-variété projective de $\mathcal{B}$. La variété $\mathcal{B}_u$ est appelée *fibre de Springer* car elle peut être vue comme la fibre au dessus de u de la résolution des singularités